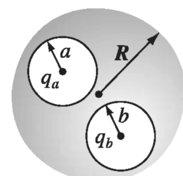
Electricity and Magnetism, Test 2, 17 March 2023

6 questions, 53 points

> Write your name and student number on each answer sheet. Use of a calculator is allowed. In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol. Before handing in: check that you read all questions carefully, and that you explained your answers where needed.

1. Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting solid sphere of radius R. At the center of each cavity a point charge is placed - call these charges q_a and q_b.



   (a) (3 points) Find the surface charge densities σ_a, σ_b, and σ_R, where the subscripts refer to the surfaces of the two cavities and surface of the solid sphere, respectively.

   (b) (2 points) What is the field inside and outside the conductor?

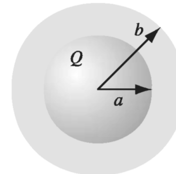
   (c) (2 points) What is the field within each cavity?

   (d) (1 point) What is the force on q_a and q_b?

This is Problem 2.39 from the book:
(a) $\sigma_a = -\frac{q_a}{4\pi a^2}$; $\sigma_b = -\frac{q_b}{4\pi b^2}$; $\sigma_R = \frac{q_a+q_b}{4\pi R^2}$
1 *point for good answer for each of the charge densities*
-0.5 *points for improper signs of each of the configurations*
(b) $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a+q_b}{r^2}\hat{\mathbf{r}}$; $\mathbf{E}_{\text{in}} = \mathbf{0}$
1 *point for noting the electric field inside and outside the conductor, respectively*
(c) $\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\hat{\mathbf{r}}_a; \mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\hat{\mathbf{r}}_b$
-0.5 *points for stating* r^2 *instead of* r_a^2 *or* r_b^2 *and not explaining that it is centered at the origin of the cavities*
(d) Zero
-0.5 *points for missing vector signs for each of the sub questions*
Common mistakes:

- *Not defining the direction vectors in c) or stating that a or b are themselves vectors, disregarding their definitions as radii.*
- *Not defining the signs of surface charge densities.*
- *Confusing conductors with dielectrics and applying the equations for bound charges.*
- *Taking the total surface charge of the sphere to be zero.*
- *Using volume instead of surface area while calculating surface charge densities.*

2. A metal sphere of radius a carries a charge Q (see figure on the right). It is surrounded, out to a radius b, by linear dielectric material of permittivity ϵ.



   (a) (3 points) Calculate the displacement $\mathbf{D}$ for all points $r > a$.
   (b) (4 points) Calculate the electric field $\mathbf{E}$ for the regions $r < a$, $a < r < b$ and $r > b$.
   (c) (4 points) Calculate the potential at the center (relative to infinity).
   (d) (3 points) Calculate the bound volume charge ρ_b in the dielectric layer.
   (e) (3 points) Calculate the bound surface charge σ_b for the inner and the outer surface of the dielectric layer.

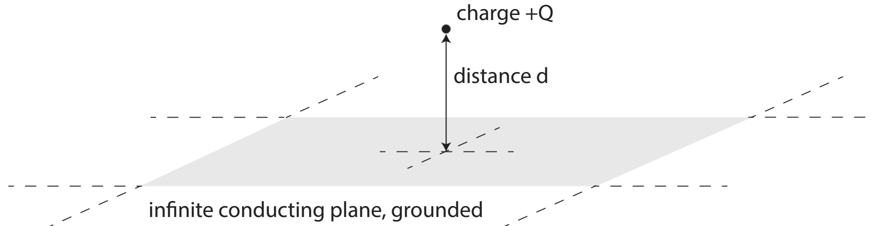
   Solution: This is example 4.5 from the book.
   (a): Calculate $\mathbf{D}$ *by using Gauss' law for the free charges. We get* $\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}$ *for all points* $r > a$*. Note there is no* ϵ_0 *in this equation!*
   (b): Inside the metal sphere ($r < a$), $\mathbf{E} = 0$*, since it is a conductor. In the dielectric* ($a < r < b$) *we find* $\mathbf{E} = \frac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}$*, and outside (for* $r > b$*) we find* $\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$.
   (c): The potential at the center can be found by integrating the electric field from infinity to the center:

$$V = -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l} = \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right)$$

   (d): The volume bound charge can be found from the polarization $\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_0 Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}$*, and then* $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = 0$.
   (e): The surface bound charge $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \frac{\epsilon_0\chi_0 Q}{4\pi\epsilon b^2}$ *at the outer surface, and* $\frac{-\epsilon_0\chi_0 Q}{4\pi\epsilon a^2}$ *at the inner surface.*

3. (*2 Bonus points*) Can you propose an arrangement of charges that has a monopole, a dipole *and* a quadrupole contribution to the potential at the same time?
   Answer: The net charge of the proposed configuration has to be nonzero, otherwise there is no monopole moment (+1 point for monopole). For the net dipole moment we need to have a set of charges with opposite sign, and the quadrupole moment is generated by a set of dipoles with opposite orientation (+1 point for dipole and quadrupole terms). An example is a positive charge in the origin, a dipole along the x-axis, and a square arrangement of charges (with alternating positive and negative sign on the corners) in the x-z plane.
   Common mistakes: Constructing an arrangement that has net charge 0, so no monopole term. Constructing a symmetric arrangement such that the dipole moment cancels. Not defining an origin in ambiguous situations. Not defining what the charges are.

4. An infinite conducting plane is grounded (the potential is zero). Above the plane, at a distance d, is a charge $+Q$.



   (a) (5 points) Explain why we can use the method of images in this case, and then use it to find the equation describing the potential between the plane and the charge.

Answer: The uniqueness theorems tell us that when the boundary conditions are known, there can be only one unique solution for the potential and electric field. Here, the boundary conditions are the plane (which is grounded, so $V = 0$), and the fact that at infinity the potential drops to zero. So an analogous situation with similar boundary conditions in the region of interest can be used to include the effect of the image charges in the plane. This is shown in the book on in section 3.2.1. For the analogous situation we imagine a charge $-Q$ situated a distance d below the plane. The potential of that situation is the same as for the situation sketched - at least in the region between the potential and the plane. We find, using the superposition principle:

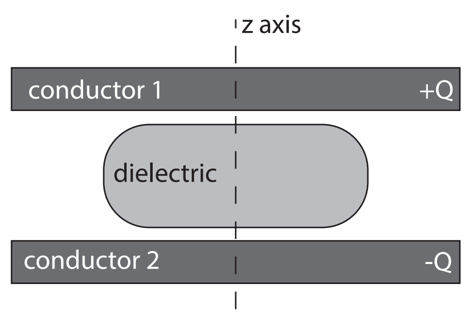
$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

(+1 point for mentioning the uniqueness theorem, +1 point for stating the boundary conditions, +3 for a correct expression of the potential)

(b) (5 points) Calculate the force between the charge and the plane. Explain how you obtain your answer. *The charge is attracted toward the plane, because of the induced negative charge. See equation (3.12) in the book. You do not need to calculate the electric field; from the analogous situation of a charge $-Q$ at a distance 2d the force can directly be found: $\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{Q^2}{(2d)^2}\hat{\mathbf{z}}$. (+2 points for correct explanation, +3 points for correct expression for the force, including the direction)*

(c) (2 points) Suppose now that the plane and the charge are completely submerged in mineral oil, which is a linear dielectric liquid. Explain what happens to the electric field everywhere. *Answer: Since the dielectric liquid is everywhere, all electric fields are reduced due to the polarization of the medium (+1 for identifying that the dielectric medium will get polarised, +1 for stating that all electric fields will be reduced as a result of the fields produced by the polarization of the dielectric medium).*

5. Consider two parallel conducting plates, each with thickness a. Their separation is d. Conductor 1 and 2 carry a charge of +Q and -Q respectively. In between the two conducting plates a neutral and solid block of a linear dielectric material is placed. The dielectric material has rounded edges. A cross-section of the situation is depicted in the figure on the right.



(a) (3 points) Copy the sketch to your answer sheet, and indicate where the free and bound (surface and/or volume) charges in this situation are located. Indicate their sign and indicate their density if it is not uniform.

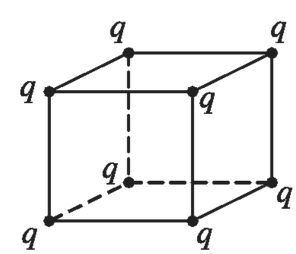
(b) (5 points) Make a qualitative graph of the potential along the z-axis. Make sure to include the region above and below both conductors. Clearly indicate for all relevant sections whether the potential is constant, or changes linearly, or has a different behaviour ($1/r$ or $1/r^2$ for example).

(c) (5 points) Make a qualitative graph of the magnitude of the z-component of the electric field along the z-axis. Make sure to include the region above and below both conductors. Clearly indicate for all relevant sections whether the magnitude of the field is constant, or changes linearly, or has a different behaviour ($1/r$ or $1/r^2$ for example). Represent the direction of the field in the sign of the magnitude.

(d) (3 points) Now while we maintain the charges on the conductors (so the conductors are not connected to a powersupply that maintains a specific potential), we remove the dielectric material. Explain what will happen to the electric field between the two conductors.

(a):The free charges are on the surfaces of the conductors - positive on conductor 1, negative on conductor 2. Most of the bound charges are on the top and the bottom of the dielectric, and their density tapers of around the rounded edges. In the central part of the capacitor the field is uniform - so there are no volume bound charges. The density of the free charges on the conductor is a bit higher near the dielectric.
(b):The potential is constant inside the conductors: positive for conductor 1, negative for conductor 2. There is a steep linear decline between the conductors and the dielectric, and a steady (but less steep) decline of the potential inside the dielectric. Outside the conductors, the potential drops off first lineraly then as $1/r$ then to $1/r^2$.
(c):The electric field is zero inside the conductors. It is constant high between the conductors and the dielectric, and constant but lower inside the dielectric. Outside the conductors the field is first constant then drops of as $1/r^2$ then $1/r^3$ (Take derivative of Potential to see).
(d):The capacitance is reduced when the dielectric material is removed. This happens because the free charge is no longer partially cancelled by the bound charges. Because the charge is kept constant, the voltage difference between the conductors will increase. Their distance is the same - so also the electric field strength will increase. 1 point mentioning the field increases, full points for complete explanation

6. (*2 Bonus points*) In this question, I ask you to justify Earnshaw's theorem: *'A charged particle can not be held in a stable equilibrium by electrostatic forces alone'.* To do this, briefly explain in words why a charge q can not be trapped in the center of a cubical arrangement of other charges (with charges q at all 8 corners) even though it seems like the repulsive forces from each corner charge could balance each other.



A stable equilibrium is a point of local minimum in the potential energy (+1 point). In our case the potential energy is qV. But we know from Laplace's equation that no local minima for V are allowed (+1 point). What looks like a minimum must in fact be a saddle point, and the box "leaks" through the center of each face.
Common mistakes:
Explaining how the particle would behave if it experiences a force. Not explaining that a stable equilibrium requires a local minimum.