# Electricity and Magnetism, Test 3, 13/04/2023

3 questions, 50 points

| |
|---|
| Write your name and student number on each answer sheet. Use of a calculator is allowed. In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol. Before handing in: check that you read all questions carefully, and that you explained your answers where needed |

1. A solenoid
   (a) (5 points) Calculate the magnetic field inside and outside an infinitely long solenoid with $n$ turns per unit length, radius $R$ and a current $I$. Explain the direction of the field using symmetries, and use Ampere's law to derive the expressions.
   (b) (5 points) Inside the solenoid from 1.(a) we place another, with the inner solenoid having $2n$ turns per unit length, and radius $\frac{1}{2}R$. The two solenoids are co-axial, and each has the same current $I$. Give the magnetic field inside, between, and outside the solenoids when the current flows in the same and opposite directions. Explain your answer.
   (c) (5 points) Consider a cylindrical paramagnetic core inside a single infinitely long solenoid, which has a current $I$ flowing through its windings. Sketch the direction of the magnetisation of this material and the direction of any volume bound current and/or surface bound current. In your sketch also include the direction of the free current in the coil.
   (d) (5 points) Let's assume the material has a linear susceptibility $\chi_m$. Find the general expression for $\mathbf{B}$ inside the material (still inside the solenoid).
   (e) (5 points) Now we consider a vertically oriented emtpy solenoid that is not infinitely long. 1) Sketch the magnetic field lines near the top of such a solenoid, 2) Explain how the force balance can lead to the levitation of a frog placed in this area, and 3) explain whether this levitation still works if the direction of the current in the solenoid is reversed.

<u>Answer:</u>
*(a): The solenoid is chosen to be oriented with its main axis in the z-direction. From symmetry arguments, the field outside is zero, and the field inside is pointing either in +z or -z, depending on the direction of the current chosen. For an infinite solenoid without material inside, the equation found from Ampere's law is* $\mathbf{B} = \mu_0 n I \hat{\mathbf{z}}$. *Outside the field is zero. The radius is not relevant.*
*(b): The magnetic fields add inside the inner solenoid. In between the solenoids there is just the field of the outer solenoid, and outside the field is still zero. Depending on the direction of the current, the direction of the field of the two coils is in the same direction or not. The radius of the solenoids is not relevant. The inner solenoid has twice the magnetic field strength compared to the outer solenoid.*
*(c): For paramagnetism, the magnetisation* $M$ *is parallel to* $B$. *A surface bound current runs*

*in the same direction as the free current, circumferential to the core material. There is no volume bound current since the magnetisation is uniform, and therefore it has no curl.*
*(d): The magnetic field is now given by* $\mathbf{B} = \mu\mathbf{H} = \mu n I\hat{\mathbf{z}} = \mu_0(1+\chi_m)nI\hat{\mathbf{z}}$
*(e): Near the top of the solenoid there is a diverging magnetic field, as in Figure 6.3 in the book. A diamagnetic object (like a frog) will be repelled by a region of strong magnetic fields, because a magnetization is induced that is opposite to the direction of the magnetic field. Comnbined with gravity, this creates a potential that is stable and can levitate the frog. If the current direction is reversed, the induced magnetization is also reversed, and the situation is still the same - levitation still works.*

2. Current loops and dipoles

   (a) (5 points) Calculate the magnetic field from a perfect magnetic dipole from its vector potential, which is $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin(\theta)}{r^2}\hat{\phi}$. Don't just give the answer: show the intermediate steps.

   (b) (5 points) Use the Biot-Savart law to show that the magnetic field a distance $z$ above the center of a circular current loop of radius $R$, which carries a steady current $I$, is given by

   $$B(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

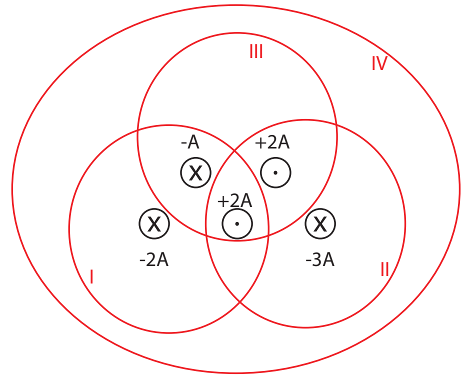
   (c) (5 points) Show that - for distances $z$ large compared to the radius of the loop - the field of the current loop is well approximated by that of a perfect magnetic dipole.

*Answer:*
*(a): Compute the curl of A corretly in spherical coordinates to get* $\mathbf{B}_{dip} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta})$.
*(b): Example 5.6. Use the Biot-Savart law to find* $B(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$
*(c): Using* $m = Ia$*, where* $a = \pi R^2$*, and the fact that* $\theta = 0$ *on the z-axis, it follows that the two equations are equal for* $z >> R$*. The* $\theta$*-component of the dipole is zero, so only the* $r$*-component remains.*

3. Amperian loops and Gaussian surfaces

Five long straight wires run perpendicular to this page, as illustrated by the black circles in the figure on the right. The wires carry currents of varying magnitude, where the sign indicates the direction of current flow: negative means current flow into the page, the positive sign means current flow out of the page. So +2A is 2 ampere out of the page, for example.

Four Amperian loops (in red, labels I, II, III and IV) are also indicated.

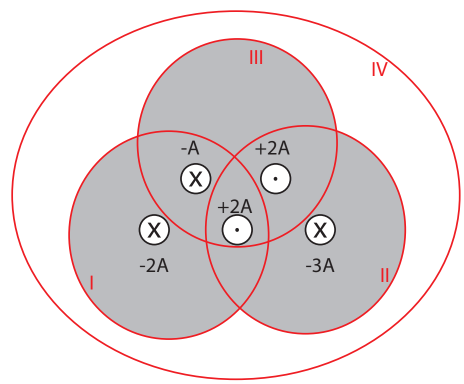



(a) (2 points) When integrating the magnetic field along each of these four loops, which one will give the largest magnitude?

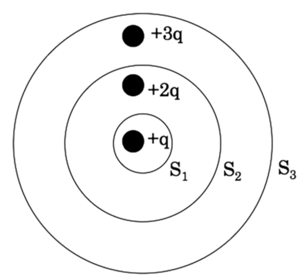
(b) (2 points) A students looks at this situation, and suggests that the sum of the magnetic field integrals along the inner loops I, II and III is always the same as the integral along outer loop loop IV, as long as they are integrated in the same direction (for example clockwise). Explain whether this is correct.

(c) (2 points) Suppose we move a charged particle with velocity $v$ in a full loop along path IV. How does the amount of work required to do this compare to the work required to move the particle, also with velocity $v$, in a full loop along path III?

(d) (2 points) Now the wires are surrounded by a paramagnetic material, up to the loops I, II and III (grey in the figure on the right). Between these loops and loop IV, the magnetic susceptibility is zero. Consider the integral of the magnetic field along path IV. Compared to the situation in question a), is its magnitude now increased, decreased or does it remain the same? Explain!




(e) (2 points) Three small positive charges ($+q, +2q, +3q$) are enclosed by three surfaces ($S_1, S_2, S_3$). A cut-through is shown in the figure on the right. The circles now represent Gaussian surfaces, not Amperian loops. The net electric flux through $S_1$ is $\Phi_E$. What is the net electric flux through $S_3$, in units of $\Phi_E$?




*Answers:*

*(a) Loop I: -$\mu_0 A$, loop II: +$\mu_0 A$, III: +3$\mu_0 A$, IV: -2$\mu_0 A$ so it is loop III.*

*(b) Because some of the wires are counted double (they are included in multiple loops) it doesn't work. This could be shown by seeing that the sum of I, II and III (=+3$\mu_0 A$) is not equal to that of IV (=-2$\mu_0 A$).*

*(c) Moving a particle in a magnetic field doesn't require any work - so it's the same*
*(d) The magnetic field is the same - because in loop IV there is no magnetization, so* $\mathbf{B} = \mu_0 \mathbf{H}$.
*Inside the grey area, the magnetic field is a bit stronger than before.*
*(e) The total flux through surface* $S_3$ *is* $6\Phi_E$