Name: Student Number:

Mechanics and Relativity: Resit R

February 14, 2023, Aletta Jacobshal

Duration: 120 mins

Before you start, read the following:

- There are 2 problems, for a total of 31 points.
- In question 1 there are 3 sub-questions (h,i and j); you have to make only one of these to get full score
- In question 2 there are 2 sub-questions (g and h); you have to make only one of these to get full score
- You can attempt all optional questions, but your finale score can never exceed a 10; if you have time to make more than 1 optional question it will only be graded / count if you have finished all non-optional questions
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

	Points
Problem 1:	16
Problem 2:	15
Total:	31
GRADE (1 + # Total/(31/9))	

Useful equations:

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

$$\Delta t \geq \Delta s \geq \Delta \tau$$

The Lorentz transformation equations with $\gamma \equiv (1-\beta^2)^{-1/2}$:

$$t' = \gamma(t - \beta\, x) \qquad x' = \gamma(x - \beta t) \qquad y' = y \qquad z' = z\,. \tag{1}$$

The relativistic Doppler shift formula and Einstein velocity transformations:

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1+v_x}{1-v_x}} \quad v'_x = \frac{v_x - \beta}{1-\beta v_x} \quad v'_{y,z} = \gamma^{-1}\frac{v_{y,z}}{1-\beta v_x}\,.$$

Possibly relevant equations:

$$F = G\frac{Mm}{r^2}; \qquad F = ma; \qquad PV \propto k_b T; \qquad F = \frac{dp}{dt}$$

Possibly relevant numbers:

$$h = 6.626 \times 10^{-34}\ \mathrm{m^2\ kg\ s^{-1}} \qquad c = 299792458\ \mathrm{m/s} \qquad 1\ \mathrm{eV} = 1.602176565 \times 10^{-19}\ \mathrm{J}$$

$$M_{\rm sun} = 1.989 \times 10^{30}\ \mathrm{kg} \qquad L_{\rm sun} = 3.83 \times 10^{26}\ \mathrm{kg\ m^2\ s^{-3}} \qquad M_{\rm gal} \sim \mathcal{O}(10^9 M_{\rm sun})$$

Question 1: In flight entertainment (16 pts)

A plane of proper length L is moving with respect to the ground with a speed $v_{\text{plane}} = 1/2$. Inside the back of the plane, some clown throws a ball towards the front of the plane with a speed of $v'_{\text{ball}} = 1/3$. For clarity, adopt unprimed coordinates (x) for the ground frame, primed coordinates (x') for the frame comoving with the plane and double primed (x'') coordinates for observers comoving with the ball.

(a) **(5 pts)** Draw a space-time diagram including the following:

- The spacetime coordinates of the ground frame
- The spacetime coordinates of the frame comoving with the plane
- The worldline of the ball
- The worldregion of the plane

Label the axis in units of L, the proper length of the plane. Assume the ground frame is the HOME frame (as defined in Moore) and the frame comoving with the plane as the OTHER frame. Assume the back of the plane is located at the origin at time $t = t' = 0$. Label the spacetime coordinate event when the clown throws the ball as **event** A and the spacetime coordinate event when it reaches the front of the plane as **event** B. Use the two observer diagram paper provided at the end of this exam.

(b) **(1 pt)** Based on your space-time diagram, as observed from the ground (HOME) frame, read off

- the time it takes for the ball to reach the front of the plane in units of L, i.e. the time elapsed between events A and B as measured on a clock at rest
- the distance the ball covers in that time in units of L, i.e. the distance between events A and B as measured on a ruler at rest

If your space-time diagram is correct, you should be able to obtain the answer with $\sim 10\%$ accuracy.

(c) **(1 pt)** Answer the same question but now for the frame comoving with the plane (OTHER).

(d) **(3 pts)** To check your answer of question (b) (HOME frame), compute the distance and time covered using velocity transformations.

(e) **(1.5 pts)** Now try to obtain the answer of question (b) and (d) by using the Lorentz transformation to go from the frame comoving with the plane to the ground frame.

(f) **(2 pts)** What is the distance and time between A and B for a ruler and a clock comoving with the ball?

(g) **(1.5 pts)** Confirm that the spacetime interval between events A and B, Δs^2, is frame independent (i.e. is the same in the ball, plane and ground frames).

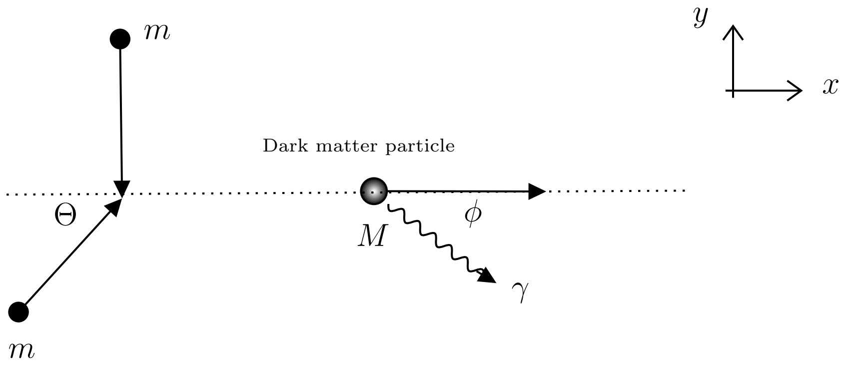
(h) **(1 pt, option 1)** Show that the time interval between events A and B as measured from the ground is related to the time interval observed from the ball with the appropriate Lorentz factor γ.

(i) **(1 pt, option 2)** Show that the time interval between events A and B as measured from the plane is also related to the time interval observed from the ball with the appropriate Lorentz factor γ

(j) **(1 pt, option 3)** Finally, is the time interval between events A and B in the ground frame and in the frame comoving with the place related by a Lorentz factor? Why or why not? Explain your answer.

Question 2: Dark matter creation (15 pts)

In this question we will look into the hypothetical process of creating a dark matter particle via the annihilation of two standard matter particles. In the process, a dark matter (DM) particle and a photon are created as shown below



Here γ presents the photon (not to be confused with the Lorentz factor), M is the mass of the created dark matter particle, the two standard model particles both have mass m. One of the particles moves along the vertical axis (y) and the other particle is moving with an angle Θ w.r.t. the horizontal axis (x). The produced DM particle moves along the horizontal axis. The photon moves with some angle ϕ w.r.t. to the horizontal axis. **We assume that the incoming particles move at the same speed and that $0 \leq \Theta, \phi \leq \pi/2$.**

(a) **(3 pts)** Write down the components of the four vectors of the particles (including the photon). You should write the components in terms of energy E and 3-momentum $\vec{p}$. Label these components and make sure you explain what the labels represent. For example, the produced DM particle has four-vector:

$$\mathbf{p}_M = (E_M, p_M, 0, 0). \tag{2}$$

Here E_M is the energy of the DM particle and p_M is the DM particle momentum in the $+x$ direction. You can use 'box' notation for the four-vectors, i.e. $\mathbf{p} = \boxed{p}$.

(b) **(1.5 pts)** Using the conservation of 4-momentum find one expression for p_M (1) and another one for E_M (2) in terms of the 3-momentum amplitude and energy E of the incoming particles p, the energy of the photon E_γ, Θ and ϕ. Find a third (3) equation relating p and Θ to E_γ and ϕ.

(c) **(1.5 pts)** Now consider $\Theta = \pi/2$. From the derived equations, explain the two possible scenarios that can happen.

(d) **(1.5 pts)** Next consider $\Theta = 0$. Explain what happens when either $\phi = 0$ or $\phi = \pi/2$.

(e) **(1.5 pts)** When does the resulting DM particle have the largest mass? Explain your answer. Show that this mass is given by

$$M = 2\gamma_m m, \tag{3}$$

where γ_m is the Lorentz factor of the two annihilating particles.

(f) **(2.5 pts)** Using dimensional analysis, derive a relation between the energy E_γ, Planck's constant h and the frequency ν of a photon . Use L, M and T as fundamental dimensions. Use information provided on the cover page.

(g) **(3.5 pts, option 1)** In our universe, the ratio of DM to ordinary matter is roughly 5 to 1 (i.e. there is about 5 times as much DM as there is ordinary matter). In the center of our galaxy, processes occur that accelerate standard model particles to 0.9998(c). Observations of the center galaxy show an excess of gamma-ray radiation with a mean frequency of $\nu \sim 10^{20}$ Hz. Assuming that the annihilating particles have a mass of $\sim$ 1 GeV, guesstimate the mass of the produced DM particle.

(h) **(3.5 pts, option 2)** Suppose the gamma-ray excess observed from the center of the galaxy equates to the luminosity of 10^7 suns, guesstimate the time it takes for all gas in the center of our Galaxy to be converted to dark matter (**assume that the gas is about 10% of the mass of ordinary matter, and that most matter in our galaxy is at the center of the galaxy**). How does this compare to the current age of the universe? You can assume a DM particle of ~ 100 GeV.

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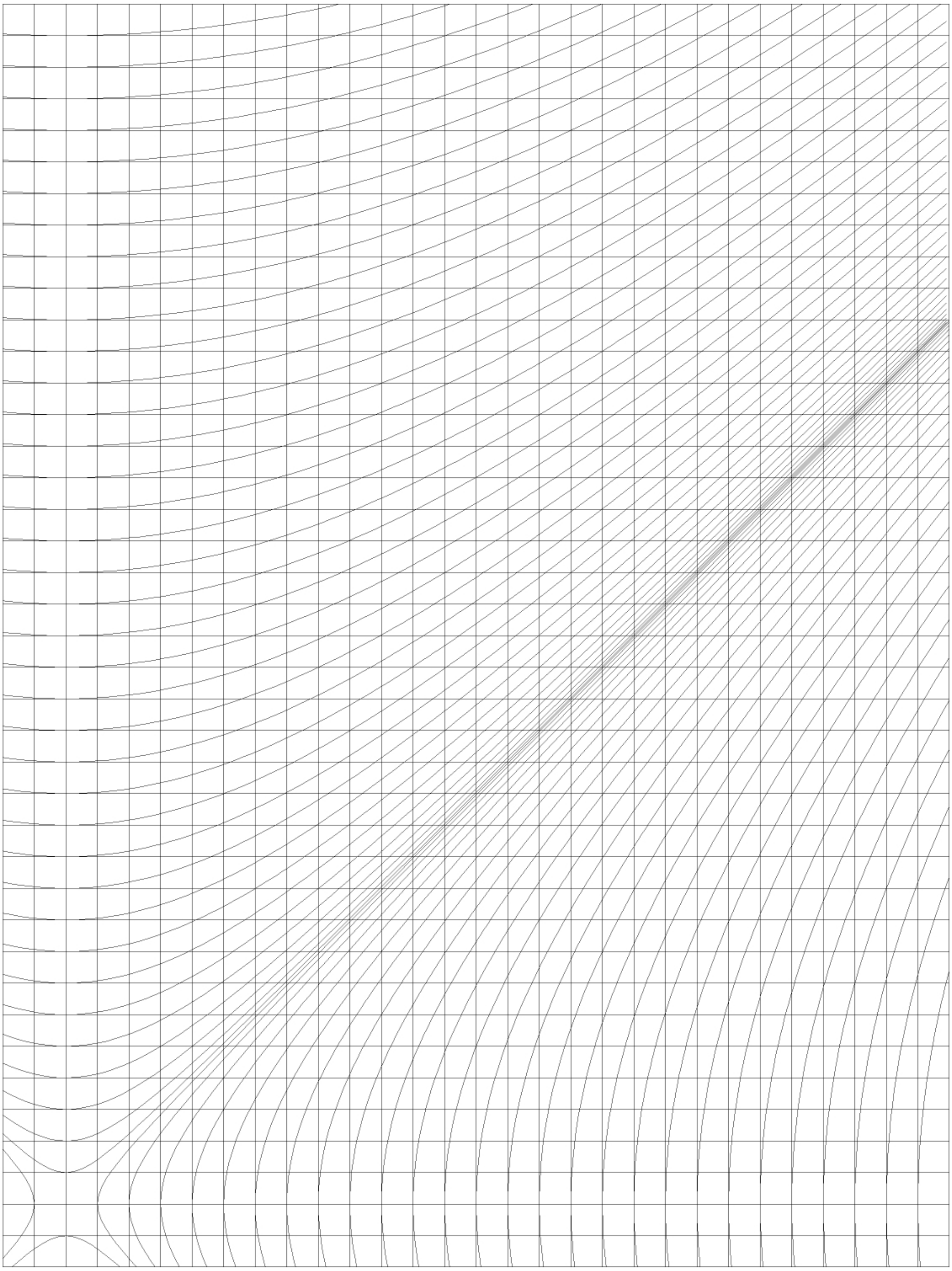

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