Mathematical Physics 2025
Exam 18 June
08:30 - 10:30

Instructions:

- Write your student number on each sheet you submit.
- You are allowed to bring one A4 cheat sheet (double-sided).
- No calculators, textbooks, or digital devices are allowed.
- Write clearly and legibly. Show all necessary steps in your calculations and clearly state any assumptions or theorems used.
- If you use a convention that is not defined in the lectures or textbooks, you must explain it clearly. Otherwise, points will be deducted.
- There are four problems in total. The total score is 100 points. This exam counts for 70% of your final grade.

Useful Identities and Equations

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sin^2 a + \cos^2 a = 1$$

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\, dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}, \quad \text{for } a > 0$$
$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \frac{d^2 u}{dx^2} = 0 \quad \text{or} \quad \nabla^2 u = 0 \quad \text{(in higher dimensions)}$$

Please **submit your exam at the front desk** according to your group, as determined by your student number in the table on the next page. Upon submission, you will find a list to sign.

Group	Student Number Range
Group 1	Student Number $<$ 5450000
Group 2	5450000 $<$ Student Number $<$ 5650000
Group 3	5659035 $<$ Student Number $<$ 5800000
Group 4	5801000 $<$ Student Number $<$ 5876000
Group 5	5876050 $<$ Student Number $<$ 5933000
Group 6	5934000 $<$ Student Number $<$ 5980000
Group 7	5990000 $<$ Student Number $<$ 6038000
Group 8	6039000 $<$ Student Number $<$ 6087000
Group 9	6087500 $<$ Student Number $<$ 6117200
Group 10	6117240 $<$ Student Number

Problem 1: (11 pts + 10 pts)

(a) Using an appropriate test, determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$$

(b) Find the radius and interval of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$$

Problem 1:

(a) Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}.$$

We apply the Alternating Series Test, which states that the series $\sum(-1)^{n-1}b_n$ converges if the sequence (b_n) satisfies the following conditions:

(i) Positivity: The terms $b_n = \frac{1}{3+5n}$ are positive for all $n \geq 1$. (1 pt for general statement, 2 pts for actually checking it)

(ii) The sequence (b_n) is decreasing:

$$b_{n+1} = \frac{1}{3+5(n+1)} = \frac{1}{8+5n} < \frac{1}{3+5n} = b_n.$$

(1 pt for general statement, 2 pts for actually checking it)

(iii) Limit to zero:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3+5n} = 0.$$

(1 pt for general statement, 2 pts for actually checking it)

Therefore, by the Alternating Series Test, the series converges.

All conditions hold since $b_n > 0$, $b_{n+1} < b_n$, and $\lim_{n\to\infty} \frac{1}{3+5n} = 0$. Therefore, the series converges. (2 pts)

(b) Using the root test(2 pts),

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n^n}\right|} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0. \quad (4pts)$$

Since $L = 0 < 1$ for all x, the radius of convergence is

$$R = \infty, \quad (2pts)$$

and the interval of convergence is

$$(-\infty, \infty). \quad (2pts)$$

Problem 2: (14 pts + 10 pts)
Consider the second-order linear differential equation:

$$y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

(a) Assume a power series solution around $x_0 = 0$. Find the recurrence relation for the coefficients a_n.

(b) Using the initial conditions, compute the first four non-zero terms of the power series. Based on this pattern, determine whether the solution is an even function, odd function, or neither. Justify your answer.

Problem 2:

(a) Assume

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \quad (3pts)$$

Substitute into the equation:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0,$$

which simplifies to

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.(3pts)$$

Re-index the first sum with $k = n - 2$:

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Rewrite the sums as

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0.$$

Separate the $k = 0$ term :

$$(k = 0): \quad (2)(1) a_2 x^0 - a_0 x^0 = 2a_2 - a_0.(2pts)$$

Then for $k \geq 1$,

$$(k+2)(k+1) a_{k+2} - k a_k - a_k = (k+2)(k+1) a_{k+2} - (k+1) a_k = 0.(2pts)$$

Thus,

$$2a_2 - a_0 = 0, (2pts)$$

and for $k \geq 1$,

$$a_{k+2} = \frac{(k+1)}{(k+2)(k+1)} a_k = \frac{a_k}{k+2}.(2pts)$$

(b) Using initial conditions $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$ (4 pts):

$$a_2 = \frac{a_0}{2} = \frac{1}{2}, \quad a_3 = \frac{a_1}{3} = 0, \quad a_4 = \frac{a_2}{4} = \frac{1}{8}, \quad a_5 = \frac{a_3}{5} = 0.$$

The first four nonzero terms are

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} \cdots .(4pts)$$

Since all odd coefficients $a_1, a_3, a_5, \ldots$ are zero, the solution is an even function (2 pts) .

Problem 3: (6 pts + 16 pts)

Let $f(t) = 1 + \sin^2 t$.

(a) Determine the fundamental period of the function $f(t)$. Determine whether $f(t)$ is an even, odd, or neither function. Justify your answer.

(b) Find the Fourier coefficients and write down the Fourier series of $f(t)$.

Problem 3:

(a) Since $\sin t$ has period 2π, and $\sin^2 t = \frac{1-\cos 2t}{2}$ has period π, the fundamental period of $f(t)$ is

$$T = \pi.(4pts)$$

To check parity:

$$f(-t) = 1 + \sin^2(-t) = 1 + \sin^2 t = f(t),$$

so $f(t)$ is an even function. (2 pts)

If a student gets the period wrong but correctly computes the Fourier expansion in part (b) based on that incorrect period, the grading should be adjusted to avoid double penalization.

(b) Using the identity

$$\sin^2 t = \frac{1 - \cos 2t}{2},$$

we rewrite

$$f(t) = 1 + \frac{1 - \cos 2t}{2} = \frac{3}{2} - \frac{1}{2}\cos 2t.(2pts)$$

The Fourier series of $f(t)$ with fundamental period π is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{\pi} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{\pi}.(4pts)$$

Because $f(t)$ is even, all $b_n = 0$ (2 pts).

From the expression above, we identify

$$a_0 = \frac{3}{2}(2pts), \quad a_1 = -\frac{1}{2}(2pts), \quad a_n = 0 \text{ for } n \neq 0, 1 (2pts).$$

Therefore, the Fourier series is

$$f(t) = \frac{3}{2} - \frac{1}{2}\cos 2t.(2pts)$$

Problem 4: (8 pts + 10 pts + 5 pts)
The time-dependent Schrödinger equation for a free quantum particle of mass m in one spatial dimension is given by

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}$$

where $\hbar$ is a constant (the reduced Planck constant) Assume $\psi(x,t)$ is defined for all $x \in \mathbb{R}$ and for $t \geq 0$, and that $\psi(x,t)$ and its Fourier transform are well-behaved.

(a) Use the Fourier transform to convert the partial differential equation into an ordinary differential equation in ω-space, and solve for $\tilde{\psi}(\omega,t)$ in terms of $\tilde{\psi}(\omega,0)$ where $\tilde{\psi}(\omega,t)$ denotes the Fourier transform of $\psi(x,t)$.

(b) Obtain the solution $\psi(x,t)$ for the following initial condition: $\psi(x,0) = \delta(x - x_0)$. Hint:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a).$$

(c) Now, consider the following two initial conditions:

$$\psi_1(x,0) = e^{-ax^2} \quad \text{for all } x \in \mathbb{R} \quad \text{(Gaussian)}$$
$$\psi_2(x,0) = \begin{cases} 1, & -L < x < L \\ 0, & \text{otherwise} \end{cases} \quad \text{(rectangular pulse)}$$

Compare the high-frequency content of the Fourier transforms of the Gaussian and the rectangular pulse. Which function contains more high-frequency content, and why? How does this difference in frequency content affect the time evolution of each wave shape? You do not need to perform any calculations—just provide a qualitative explanation.

Problem 4:

(a) The Fourier transform of $\psi(x,t)$ is defined as

$$\tilde{\psi}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x,t)e^{-i\omega x}\,dx.(2pts)$$

Applying the Fourier transform to the Schrödinger equation,

$$i\hbar\frac{\partial\tilde{\psi}(\omega,t)}{\partial t} = \frac{\hbar^2\omega^2}{2m}\tilde{\psi}(\omega,t).(2pts)$$

by using

$$\mathcal{F}\left[\frac{\partial^2\psi(x,t)}{\partial x^2}\right](\omega) = (i\omega)^2\tilde{\psi}(\omega,t) = -\omega^2\tilde{\psi}(\omega,t).(2pts)$$

This ordinary differential equation has the solution

$$\tilde{\psi}(\omega,t) = \tilde{\psi}(\omega,0)\,e^{-i\frac{\hbar\omega^2}{2m}t}.(2pts)$$

(b) The Fourier transform of the initial condition $\psi(x, 0) = \delta(x - x_0)$ is calculated as follows:

$$\tilde{\psi}(k, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - x_0) e^{-ikx}\, dx . (2pts)$$

Using the hint, this evaluates to

$$\tilde{\psi}(k, 0) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0} . (2pts)$$

Therefore,

$$\tilde{\psi}(k, t) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0} e^{-i\frac{\hbar k^2}{2m}t} . (2pts)$$

The inverse Fourier transform gives the solution in position space:

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(k, t) e^{ikx}\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x - x_0)} e^{-i\frac{\hbar k^2}{2m}t}\, dk . (2pts)$$

Evaluating the Gaussian integral yields the free particle propagator:

$$\psi(x, t) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(\frac{im(x - x_0)^2}{2\hbar t}\right) . (2pts)$$

(c) (2 points for stating that the pulse contains more high-frequency components, and 1 point for explaining that this is due to its sharp discontinuities.
2 points for explaining that the Gaussian wave packet spreads smoothly and largely retains its shape over time, while the pulse—requiring many high-frequency modes to represent its sharp edges—undergoes more significant shape changes as these high-frequency components decay more rapidly.)