Linear Algebra - Final Exam

April 1, 2025 11.45-13.45

IMPORTANT:
At the end place your exam in the pile, **under the envelope** corresponding to your tutorial group
Do not cover the envelope with your exam, so that others can read the tutorial group number on the envelope
To avoid delays when placing the exams, make sure you **know your tutorial group number in advance** (see BS)
To avoid confusion in case exams get mixed up in the piles, write your name and student number on all pages
Number the pages of the exam sequentially
Exams not returned to the correct pile or pages without name/student number may not get graded

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Exam rules:
- You can have a "cheat sheet". This is an A4 paper written on one side (see above).
- You are NOT allowed to have books, course notes, homework assignments, etc., laptops, e-readers, tablets, telephones, etc.
- You can use a normal calculator (not a programmable/graphic one).
- Give a clear explanation of your answer and show any relevant computations.
- You get no points for a result without any calculation/explanation.
- If you're asked explicitly to use a certain method, you'll get no points if you use a different one even if your answer is right.

QUESTIONS:

1. **2.5** Given the vectors $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2\sqrt{3} \\ 1+\sqrt{3} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 2(\sqrt{3}-1) \end{bmatrix}$, there is a linear transformation, $L(\mathbf{x})$ from $\mathbb{R}^2$ to $\mathbb{R}^2$, that applied to, respectively, the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ give, respectively, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

   (a) 0.6 Find the matrix A of this transformation.

   (b) 0.6 Find the eigenvalues of A

   (c) 0.7 Find the eigenvectors of A.

   (d) 0.6 Use the eigenvalues and eigenvectors to calculate A^2.
   Note: You do not get points for this part if you do the product A times A directly.

2. **1.8** Let P_3 be the vector space consisting of all polynomials, p, with real coefficients of degree less than 3. Let $B = \{x^2, x, 1\}$ be a basis of the vector space P_3.
   For each transformation $L: P_3 \to P_3$ defined below, show that the transformation is linear and find the matrix representation of L with respect to the basis B. Finally, apply the transformation using the matrix to the polynomial $p(x) = ax^2 + bx + c$ written as the coordinate vector with respect to the basis B, and show that this is the same as calculating the derivatives/integrals to the polynomial $p(x)$ in the usual way.

   (a) 0.9 $L(p) = \left(\frac{d^2}{dx^2} - 3\frac{d}{dx}\right) p(x)$

   (b) 0.9 $L(p) = e^x \frac{d}{dx}(e^{-x} p(x))$.

3. **2** Find the **real** solution of the following initial-value problem:
   $$\begin{aligned} x_1' &= 2x_1 + x_2 + x_3 \\ x_2' &= -2x_3 \\ x_3' &= 2x_2 \end{aligned}$$
   with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

Please turn over

4. **1.3** The vectors **a**, **b** and **c** are given by $\mathbf{a} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \\ \alpha \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, with parameter α to be determined.

   (a) 0.2 For which value(s) of α is the angle between **a** and **b** equal to $\frac{3}{4}\pi$?

   (b) 0.3 For which value(s) of α are **a**, **b** and **c** linear dependent? Express **a** as a linear combination of **b** and **c** in this case.

   (c) 0.2 Using the value(s) of α from part (a), write the unit vectors in the direction of each of the three given vectors.

   (d) 0.2 For which value(s) of α are the vectors **a** and **b** orthogonal?

   (e) 0.2 For which value(s) of α are the vectors **b** and **c** orthogonal?

   (f) 0.2 Use the cross product to find a vector perpendicular to both **a** and **c**.

5. **2.4** Check whether the following are eigenvalues and eigenvectors of a matrix. Justify your answer. If the answer is yes, give the matrix.

   (a) 1.2 $\lambda_1 = \frac{5}{3}$, $\lambda_2 = 3$, $\lambda_3 = -1$, $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 5 \\ -9 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 5 \\ 8 \end{bmatrix}$.

   (b) 1.2 $\lambda_1 = -4$, $\lambda_2 = 2$, $\lambda_3 = 2$, $\lambda_4 = 1$, $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -6 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

NOTE: Maximum points possible, $p = 10$. Grade, $g = 0.9p + 1$

Solutions

As stated in the exam rules, they get no points if they give the result without explanation or calculations, even if the result is correct

1. (a) To find the matrix I need to see what the transformation of the unit vectors is, and use those as the columns of A. 0.2 points for the method

It is given that: $L\left(\begin{bmatrix}0\\2\end{bmatrix}\right) = 2L(\mathbf{e}_2) = \begin{bmatrix}-4\\2(\sqrt{3}-1)\end{bmatrix} \implies L(\mathbf{e}_2) = \begin{bmatrix}-2\\\sqrt{3}-1\end{bmatrix}$,

0.2 points for the result

and:

$$L\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = 2L(\mathbf{e}_1) + L(\mathbf{e}_2) = \begin{bmatrix}2\sqrt{3}\\1+\sqrt{3}\end{bmatrix} \implies 2L(\mathbf{e}_1) + \begin{bmatrix}-2\\\sqrt{3}-1\end{bmatrix} = \begin{bmatrix}2\sqrt{3}\\1+\sqrt{3}\end{bmatrix} \implies L(\mathbf{e}_1) = \begin{bmatrix}\sqrt{3}+1\\1\end{bmatrix}.$$

0.2 points for the result

Then:

$$A = \begin{bmatrix}\sqrt{3}+1 & -2\\ 1 & \sqrt{3}-1\end{bmatrix}.$$

(b) To find the eigenvalues solve $\det A - \lambda I = 0 \implies \begin{vmatrix}\sqrt{3}+1-\lambda & -2\\ 1 & \sqrt{3}-1-\lambda\end{vmatrix} = 0 \implies$

0.2 points for the method

$(\sqrt{3}+1-\lambda)(\sqrt{3}-1-\lambda)+2 = 0 \implies \lambda^2 - 2\sqrt{3}\lambda + 4 = 0 \implies \lambda_1 = \sqrt{3}+i, \lambda_2 = \sqrt{3}-i.$

0.4 points for the result (0.2 points per eigenvalue)

It is also okay if they find one eigenvalue using this method, and get the other as the complex conjugate of that one.

(c) To find the eigenvectors, for each eigenvalue λ_i I need to solve $(A-\lambda I)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$. For $\lambda_1 = \sqrt{3}+i$:

0.3 points for the method

$$\begin{bmatrix}\sqrt{3}+1-(\sqrt{3}+i) & -2\\ 1 & \sqrt{3}-1-(\sqrt{3}+i)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Operating on the augmented matrix (I do not show the steps) I find that:

$\mathbf{v}_1 = t\begin{bmatrix}i+1\\1\end{bmatrix}$ for $t \in \mathbb{C}$. Take $t = 1$ so that $\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix} + i\begin{bmatrix}1\\0\end{bmatrix}$.

I don't need to calculate the other eigenvector because I know it will be the complex conjugate of $\mathbf{v}_1$. Then:

$$\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix} - i\begin{bmatrix}1\\0\end{bmatrix}.$$

0.4 points for the result (0.2 points for each eigenvector)

It is correct both if they calculate only one eigenvector as I did, and take the other one as the complex conjugate, or if they calculate both step by step

(d) For this I need to use $P = [\mathbf{v}_1\ \mathbf{v}_2]$ and D a diagonal matrix with the eigenvalues along the diagonal, calculate P^{-1} and then use that $A^2 = PD^2P^{-1}$, where D^2 is simply a diagonal matrix with the square of the eigenvalues along the diagonal. To calculate P^{-1} row operate on $(P|I)$ to get $(I|P^{-1}$. I do not write down the steps. The result is:

$$P^{-1} = \begin{bmatrix}-i/2 & 1/2+i/2\\ i/2 & 1/2-i/2\end{bmatrix}.$$

0.3 points for P^{-1}. They need to explain the steps they used to do this.

Then

$$A^2 = \begin{bmatrix} i+1 & -i+1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (\sqrt{3}+i)^2 & 0 \\ 0 & (\sqrt{3}-i)^2 \end{bmatrix} \begin{bmatrix} -i/2 & 1/2+i/2 \\ i/2 & 1/2-i/2 \end{bmatrix} = \begin{bmatrix} 2+2\sqrt{3} & -4\sqrt{3} \\ 2\sqrt{3} & 2-2\sqrt{3} \end{bmatrix}.$$

0.3 points for the method of getting A^2.

They do not get the 0.3 points for A^2 if they do not use the eigenvalues and eigenvectors and, for instance, calculate $A^2 = A.A$. This was explicitly stated in the exam

2. (a) Find the effect of the transformation upon the unit vectors $\{x^2, x, 1\}$:

0.1 points for the method

$L(x^2) = 2 - 6x = 0(x^2) - 6(x) + 2(1)$
$L(x) = -3 = 0(x^2) + 0(x) - 3(1)$
$L(1) = 0 = 0(x^2) + 0(x) + 0(1).$

If they do it this way, they get 0.2 points for each of the above (0.6 points in total).

So $A = \begin{bmatrix} 0 & 0 & 0 \\ -6 & 0 & 0 \\ 2 & -3 & 0 \end{bmatrix}$.

0.2 points for the matrix

But if they do not write the matrix, or they do not write it correctly, they loose the last 0.2 points.

If they do it in a different way, they get 0.6 points for getting the matrix. They have to explain how they got it, and the procedure has to be right.

(b) We first write:

$$L(p) = e^x \frac{d}{dx}(e^{-x}p(x)) = e^x \left(-e^{-x}p(x) + e^{-x}\frac{d}{dx}p(x)\right)$$
$$= -p(x) + \frac{d}{dx}p(x).$$

0.1 points for this, or for getting this in some other correct way such that they can proceed to the next point.

Then:

$L(x^2) = -x^2 + 2x = -1(x^2) + 2(x) + 0(1)$

$L(x) = -x + 1 = 0(x^2) - 1(x) + 1(1)$

$L(1) = -1 = 0(x^2) + 0(x) - 1(1)$

If they do it this way, they get 0.2 points for each of the above (0.6 points in total).

So $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

0.2 points for the matrix.

If they do not write the matrix, or they do not write it correctly, they loose 0.2 points.

If they do it in a different way, they get the full 0.6 points for getting the matrix. They have to explain how they got it, and the procedure has to be right.

Notice that, on way or the other, the maximum points for this item is 0.6.

3. The system can be written in matrix form as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

I first find the evals/evects of A (I do not show the steps, but the students have to).

The evals are $\lambda_1 = 2$, $\lambda_2 = 2i$, $\lambda_3 = -2i$, and the corresponding events are: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -i \\ 2i \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} i \\ -2i \\ 2 \end{bmatrix}$.

0.9 for the 3 evals and 3 evects; 0.1 points for each eval and 0.2 points for each evect.

It is okay if they calculate only $\mathbf{v}_1$ and $\mathbf{v}_2$, and say that $\mathbf{v}_3$ must be the complex conjugate of $\mathbf{v}_2$

.

At this point they may go as I do below, or they can decide to change variables to decouple the equations as I explained in the class (see my notes). Both ways are fine. if they choose to decouple the equations, at the end they **must** transform back to the original variables.

I write $\lambda_2 = a + ib$, where $a = 0$ and $b = 2$.
The general **real** solution will then be:

They are allowed to take the formula directly from the book.

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \left[\mathbb{R}e(\mathbf{v}_2) \cos bt - \mathbb{I}m(\mathbf{v}_2) \sin bt\right] e^{at} + c_3 \left[\mathbb{R}e(\mathbf{v}_2) \sin bt + \mathbb{I}m(\mathbf{v}_2) \cos bt\right] e^{at}.$$

It is also possible to write the general solution $\mathbf{x} = \sum_{i=1}^{3} c_i \mathbf{v}_i e^{\lambda_i t}$, where λ_i and $\mathbf{v}_i$ are possibly complex, and keep only the real part of the solution. In all cases the results should be the same, but some of the intermediate steps may differ from what I have below.

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \cos 2t - \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \sin 2t \right) + c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \sin 2t + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \cos 2t \right) = \begin{bmatrix} c_1 e^{2t} + c_2 \sin 2t - c_3 \cos 2t \\ -2c_2 \sin 2t + 2c_3 \cos 2t \\ 2c_2 \cos 2t + 2c_3 \sin 2t \end{bmatrix}.$$

0.6 points for the general solution.

Applying the initial condition:

$$\begin{bmatrix} c_1 - c_3 \\ 2c_3 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

I can solve this to find $c_1 = 3$, $c_2 = 1$, $c_3 = 1$. The solution of the initial-value problem is then:

0.3 points for the c_i, 0.1 points for each.

$$\mathbf{x}(t) = \begin{bmatrix} 3e^{2t} + \sin 2t - \cos 2t \\ -2 \sin 2t + 2 \cos 2t \\ 2 \cos 2t + 2 \sin 2t \end{bmatrix}.$$

0.2 points for the solution of the initial-value problem.

4. (a) I know that $\mathbf{a}^T \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$

   0.1 points for this or similar statement. They get the points if they use this without stating it explicitly

   $|\mathbf{a}| = \sqrt{8}$, $|\mathbf{b}| = \sqrt{10 + \alpha^2}$ and $\mathbf{a}^T \mathbf{b} = -8 \rightarrow \cos \theta = \dfrac{\mathbf{a}^T \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, and after doing the algebra I find $\alpha = \sqrt{6}$.

   0.1 points for α

   (b) For this I can for instance calculate $\det [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \det \begin{bmatrix} 2 & -3 & 2 \\ -2 & 1 & 2 \\ 0 & \alpha & 1 \end{bmatrix}$ and find for which value(s) of α this is different from 0.

   0.1 points for the this or similar way of finding that the vectors are linearly independent

   I do not write the steps (the students have to), and find that as long as $\alpha \neq -1/2$ the vectors are linearly dependent.

   0.1 points for this statement

   (c) Taking $\alpha = \sqrt{6}$ the three vectors are:

$\mathbf{a} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \\ \sqrt{6} \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. The moduli of the three vectors are $|\mathbf{a}| = \sqrt{8}$, $|\mathbf{b}| = 4$ and $|\mathbf{c}| = 3$.

The unit vectors are:

$\mathbf{a}/|\mathbf{a}| = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{b}/|\mathbf{b}| = \frac{1}{4} \begin{bmatrix} -3 \\ 1 \\ \sqrt{6} \end{bmatrix}$, $\mathbf{c}/|\mathbf{c}| = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

0.3 points for the 3 units vectors (0.1 points for each

They get 0.1 points if they state correctly how to find the unit vectors even if they do not do it in the end, or make algebraic errors in the calculations

(d) Two vectors are orthogonal $\Leftrightarrow$ the dot product is equal to zero.

They get 0.1 points if they state this correctly; they get the 0.1 points if they do not say this explicitly but use it to find the solution

$\mathbf{a}.\mathbf{b} = -6 - 2$ which is different from zero. So **a** and **b** are never orthogonal.

0.1 points for this result

In this and the next point, they can use a different way as long as it is correct, and should get full points if they do that

(e) $\mathbf{a}.\mathbf{b} = -6 + 2 + \alpha = 0 \rightarrow \alpha = 4$.

0.2 points for this; use the same criterium as in (d) to give points

(f) $\mathbf{z} = \mathbf{a} \times \mathbf{c}$ is perpendicular both to $\mathbf{a}$ and $\mathbf{c}$.

$$z = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & -2 & 0 \\ 2 & 2 & 1 \end{vmatrix} = \mathbf{e}_1(-2) - \mathbf{e}_2(2) + \mathbf{e}_3(8) = -2\mathbf{e}_1 - 2\mathbf{e}_2 + 8\mathbf{e}_3 = \begin{bmatrix} -2 \\ -2 \\ 8 \end{bmatrix}.$$

0.2 points for the result of the cross product. They get 0.1 points if they write correctly how to calculate the cross product even if they do not compute it in the end, or make algebraic errors in the calculations

5. (a) In this case it is easy to see that the vectors are not linearly independent because, if I call $P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$, $\det P = 0$.

0.6 points for noticing that if $\det P = 0$ is enough to show no independence

0.3 points for calculating the determinant. They get the full point also if they use a different, valid, method to prove that the vectors are not independent

Then P^{-1} does not exist, and I won't be able to calculate $A = PDP^{-1}$. So the answer is no, these are not the eigenvalues and eigenvectors of a matrix.

0.3 points for the answer

(b) In this case the vectors are linearly independent. To see this I call $P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$, which is a block diagonal matrix. It is easy to calculate the determinant of the two (2×2) blocks and multiply them by each other to get that $\det P = \det \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ -6 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix} = 6.$

0.2 points for the determinant

So the answer is yes, these are the eigenvalues and eigenvectors of a matrix.

0.2 points for noticing that if $\det P \neq 0$ is enough to show independence

To find this matrix, A, I need to invert P, so I can write $A = PDP^{-1}$ where D is a diagonal matrix with the eigenvalues in the diagonal. To invert P I take advantage from the fact that P is block diagonal. The blocks, and corresponding inverses are:

$P_{11} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$, $P_{22} = \begin{bmatrix} -6 & 0 \\ 5 & 1 \end{bmatrix}$, $P_{11}^{-1} = \begin{bmatrix} -1/6 & 0 \\ 5/6 & 1 \end{bmatrix}$, and $P_{22}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$, and hence:

$$P^{-1} = \begin{bmatrix} 0 & 0 & -1/6 & 0 \\ 0 & 0 & 5/6 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}.$$

0.5 points for the inverse of P

Then:

$$A = PDP^{-1} = \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ -6 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1/6 & 0 \\ 0 & 0 & 5/6 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 5 & 2 \end{bmatrix}$$

0.3 points for A