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General Relativity 2025-2026: Assignment 1

posted at September 16, to be submitted on September 23

N.B. you can obtain a maximum of **(200pt)**

Problem 1: Indices (30pt)

a. Indicate in each of the following expressions whether the index α is a free index or a dummy index. (15pt)

$$a^\alpha b_\alpha = 1\,, \quad a^\alpha b^\beta c_\beta = d^\alpha\,, \quad a^\alpha b^\beta = d^{\alpha\beta}\,. \tag{1}$$

a

i) $a^\alpha b_\alpha = 1$ the index α is repeated both up and down

→ according to Einstein's summation convention it means that α is summed over all its possible values

→ α is a dummy index

ii) $a^\alpha b^\beta c_\beta = d^\alpha$

- as in the case above β is once up and once down → β is a dummy index
- α appears once in both sides of the equation

→ α is not summed over → α is a free index

iii) $a^\alpha b^\beta = d^{\alpha\beta}$ both α and β appear once in both sides of the equation and they do not come in the pair up - down

→ α and β are free indexes

b

b. Indicate in which of the following expressions the Einstein summation convention is used. (15pt)

$$\overset{1}{a^\alpha b_\beta = c^\alpha{}_\beta}\,, \quad \overset{2}{a^\alpha b_\alpha = 1}\,, \quad \overset{3}{c^\alpha{}_\alpha = 1}\,. \tag{2}$$

1) $a^\alpha b_\beta = c^\alpha_\beta$ → α and β only appear once per side, and no index is repeated in the same term

→ no summation convention is used!

2) $a^\alpha b_\alpha = 1$ → α is used once up and once down → it means summation over all values of α.

→ summation convention is used!

3) $c^\alpha_\alpha = 1$ → α is used again once up and once down

→ summation convention is used!

Problem 2: Scalar product (30pt)

a. Consider a spacetime with coordinates (t, x) and line element

$$ds^2 = -dt^2 + dtdx + dx^2 \,. \quad (3)$$

Write down the metric tensor components $g_{\alpha\beta}$ corresponding to this line element. (10pt)

a

line element : $ds^2 = -dt^2 + dt\,dx + dx^2$

using the fact that I can rewrite this using the quadratic formula I get:

$$ds^2 = g_{tt}\,dt^2 + 2g_{tx}\,dt\,dx + g_{xx}\,dx^2$$

I notice that since the metric is simmetric : $g_{tx} = g_{xt}$

now I match the coeff. term by term:

$dt^2 \rightarrow g_{tt} = -1$

$dt\,dx \rightarrow 2g_{tx} = 1 \rightarrow g_{tx} = g_{xt} = \frac{1}{2}$

$dx^2 \rightarrow g_{xx} = 1$

$$\rightarrow (g_{\alpha\beta}) = \begin{pmatrix} g_{tt} & g_{tx} \\ g_{xt} & g_{xx} \end{pmatrix} = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

b. Consider in this spacetime a vector

$$V^\alpha = \begin{pmatrix} z \\ 1 \end{pmatrix} \quad (4)$$

with z a real number. Calculate the scalar product $\vec{V} \cdot \vec{V}$. (10pt)

b

$V^\alpha = \begin{pmatrix} z \\ 1 \end{pmatrix} \rightarrow V^t = z \,,\ V^x = 1 \,,\ z \in \mathbb{R}$

$\vec{V} \cdot \vec{V} = g_{\alpha\beta} V^\alpha V^\beta = g_{tt} (V^t)^2 + 2g_{tx} V^t V^x + g_{xx} (V^x)^2$ (scalar product)

now I substitute the values of the components:

$$\vec{V} \cdot \vec{V} = (-1) z^2 + 2\left(\frac{1}{2}\right) z\,(1) + (1)(1)^2 = -z^2 + z + 1$$

c. For which values of z is the vector $\vec{V}$ spacelike, lightlike or timelike? (10pt)

since $g_{tt} = -1$, $g_{xx} = 1$, I consider the $(-+)$ form:

$\vec{V} \cdot \vec{V} < 0 \rightarrow$ timelike
$\vec{V} \cdot \vec{V} = 0 \rightarrow$ lightlike
$\vec{V} \cdot \vec{V} > 0 \rightarrow$ spacelike

considering the equation:

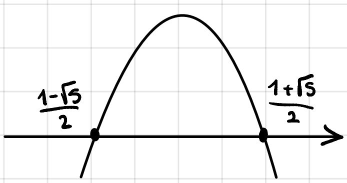
$$\vec{V} \cdot \vec{V} = -z^2 + z + 1$$

I first find the zeros:

$$-z^2 + z + 1 = 0 \leftrightarrow z^2 - z - 1 \; 0 \longrightarrow z = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$z_- = \frac{1-\sqrt{5}}{2}, \quad z_+ = \frac{1+\sqrt{5}}{2}$$

since the sign of the z^2 is negative the parabola associated with the equation opens downward:



, so:

$\vec{V} \cdot \vec{V} < 0 \rightarrow$ timelike $\rightarrow$ $z < z_-$ or $z > z_+$

$\vec{V} \cdot \vec{V} = 0 \rightarrow$ lightlike $\rightarrow$ $z = z_\pm$

$\vec{V} \cdot \vec{V} < 0 \rightarrow$ spacelike $\rightarrow$ $\frac{1-\sqrt{5}}{2} < z < \frac{1+\sqrt{5}}{2}$

Problem 3: Calculating Distances and Areas (20pt)

Consider a spacetime geometry with coordinates (t, r, θ, ϕ) and line element

$$ds^2 = -(1-Ar^2)^2\,dt^2 + (1-Ar^2)^2\,dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (5)$$

for certain constant A.

a. Calculate the proper distance along a radial line at constant t from the centre $r = 0$ to a coordinate radius $r = R$. (10pt)

if the radial curve has constant t $\rightarrow$ θ, ϕ also will be constant:

$\hookrightarrow dt = d\theta = d\phi = 0$

substituting in the line element, I get:

$$ds^2 = (1-Ar^2)^2\,dr^2$$

proper length ℓ along this curve is:

$$\ell = \int_0^R \sqrt{ds^2} = \int_0^R |1-Ar^2|\,dr$$

I use absolute value because a distance element cannot be negative

1) $A \leq 0$:

$1 - Ar^2 = 1 + |A| r^2 \geq 0$

$$\ell = \int_0^R (1-Ar^2)\,dr = \left[r - \frac{A}{3} r^3\right]_0^R = R - \frac{A}{3}R^3$$

2) $A > 0$, $R \leq r_c = 1/\sqrt{A}$

$$\ell = R - \frac{A}{3}R^3$$

3) $A > 0$, $R > r_c = 1/\sqrt{A}$

$$\ell = \int_0^{r_c} (1-Ar^2)\,dr + \int_0^R (Ar^2-1)\,dr$$

$\int (1-Ar^2)\,dr = r - \frac{A}{3}r^3$, $\int (Ar^2-1)\,dr = \frac{A}{3}r^3 - r$

$\hookrightarrow \ell = 2\left(r_c - \frac{A}{3}r_c^3\right) + \left(\frac{A}{3}R^3 - R\right)$

substitute $r_c = 1/\sqrt{A}$

$$\ell(R) = \frac{4}{3\sqrt{A}} + \frac{A}{3}R^3 - R$$

b. Calculate the area of a sphere of coordinate radius $r = R$. (10pt)

$$d\sigma^2 = R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$\hookrightarrow h_{ab} = \mathrm{diag}\left(R^2, R^2\sin^2\theta\right), \quad \sqrt{\det h} = R^2 \sin\theta$$

$$A(R) = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta\, d\theta\, d\phi = 4\pi R^2$$

Problem 4: Local lightcones (60pt)

Consider a spacetime geometry with coordinates (t, x, y, z) and line element

$$ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2) \tag{6}$$

for constant H.

a. Determine the non-zero metric components and calculate its determinant. (10pt)

a) the metric tensor $g_{\alpha\beta}$ is a 4×4 symmetric matrix

$\hookrightarrow$ we can write the line element as: $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$

comparing the 2 expressions:

$g_{00} = -1$

$g_{11} = g_{22} = g_{33} = e^{2Ht}$ non zero components

in matrix form:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{+2Ht} & 0 & 0 \\ 0 & 0 & e^{+2Ht} & 0 \\ 0 & 0 & 0 & e^{+2HT} \end{pmatrix}$$

$$g_{\alpha\beta} = \mathrm{diag}\left(-1, e^{+2HT}, e^{+2HT}, e^{+2HT}\right)$$

the determinant of a digonal matrix is the product of the element of the digonal:

$$\det(g_{\alpha\beta}) = -1\cdot\left(e^{2Ht}\right)^3 = -e^{6Ht}$$

b. Calculate the light ray trajectories in this geometry and indicate them in a spacetime diagram. *Hint:* Assume fixed (y, z) coordinates and restrict to a two-dimensional (t, x). (20pt)

light rays move along curves where $ds^2 = 0$.

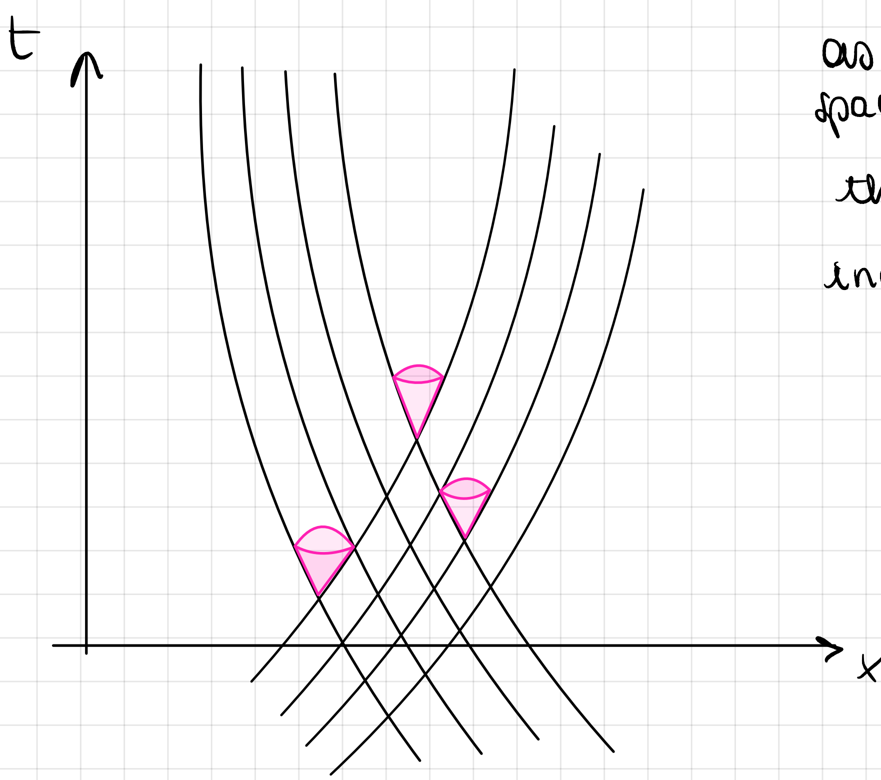
assuming fixed (x, y), we have $dy = dz = 0$

$$ds^2 = -dt^2 + e^{2ut} dx^2 = 0$$

$$\frac{dx^2}{dt^2} = e^{-2uT} \rightarrow \frac{dx}{dt} = \pm e^{-ut}$$

$$x(t) = \pm \int e^{ut} dt = \pm \frac{1}{u} e^{-ut} + c$$

c. Show, by indicating a few local lightcones in the (t, x) spacetime diagram, that these lightcones become more narrow with increasing time. (10pt)



as we can see in the space-time diagram

the light cones become narrow

increasing t

d. Find a coordinate transformation $t = t(\eta)$ such that

$$ds^2 = \Omega^2(\eta)\Big(-d\eta^2 + dx^2 + dy^2 + dz^2\Big) \qquad (7)$$

for some $\Omega(\eta)$. Also write down $\Omega(\eta)$ explicitly. (20pt)

We want a coord. transformation $t = t(\eta)$

such that: $ds^2 = \Omega^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2)$

from point b:

$$ds^2 = -dt^2 + e^{2Ht}dx^2$$

equating the 2 equations I get:

$$-dt^2 + e^{2Ht}dx^2 = \Omega^2(\eta)(-d\eta^2 + dx^2)$$

where I consider only (t, x)

$$-dt^2 + e^{2Ht}dx^2 = -\Omega^2(\eta)d\eta^2 + \Omega^2(\eta)dx^2$$

where $t = t(\eta)$

$$\begin{cases} e^{2Ht} = \Omega^2(\eta) \\ dt^2 = \Omega^2(\eta)d\eta^2 \end{cases} \longrightarrow dt^2 = e^{2Ht}d\eta^2$$

$$d\eta = e^{-Ht}dt$$

$$\eta(t) = e^{-Ht}dt \rightarrow \eta(t) = -\frac{1}{H}e^{-Ht} + \text{const}$$

$$-H\eta(t) = e^{-Ht} \rightarrow -Ht = \ln(-H\eta)$$

$$\Omega(\eta) = e^{Ht} = e^{-\ln(-H\eta)} = \frac{1}{e^{\ln(-H\eta)}} = -\frac{1}{H\eta}$$

$$\Omega(\eta) = -\frac{1}{H\eta}$$

Problem 5: Christoffel Symbols (60pt)

Consider a three-dimensional spacetime with coordinates $x^\alpha = (t, r, \phi)$ and line element:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2 . \tag{8}$$

a. Show that the Lagrangian for the variational principle for geodesics $x^\alpha(\sigma)$ in this spacetime is given by (10pt)

$$L(\dot t, \dot r, \dot\phi, r) = \left[\left(1 - \frac{2M}{r}\right)\dot t^2 - \left(1 - \frac{2M}{r}\right)^{-1} \dot r^2 - r^2 \dot\phi^2\right]^{1/2} \tag{9}$$

with $\dot x^\alpha = dx^\alpha/d\sigma$.

a) $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\phi^2$

is the line element

$$g_{\alpha\beta} = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-1} & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$ metric tensor

for the variational principle, we have to extremize the proper time between 2 points

$\tau_{AB} = \int_A^B d\tau$

$d\tau^2 = -ds^2$

$\tau_{AB} = \int_A^B \left[\left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2 d\phi^2\right]^{\frac{1}{2}}$

parametrizing the curve using σ:

$\tau_{AB} = \int_0^1 d\sigma \left[\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\sigma}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{d\phi}{d\sigma}\right)^2\right]^{\frac{1}{2}}$

where:

$\sigma = 0$ At A $\quad\quad$ $\sigma = 1$ at B

and

$\tau_{AB} = \int_0^1 L\, d\sigma$

the Lagrangian is:

$L = \left[\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\sigma}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{\partial\phi}{\partial\sigma}\right)^2\right]^{\frac{1}{2}}$

$L = \left[\left(1-\frac{2M}{r}\right)\dot t^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot r^2 - r^2\dot\phi^2\right]^{\frac{1}{2}}$

with $\dot x^\alpha = \frac{dx^\alpha}{d\sigma}$

b. Vary the Lagrangian (9) with respect to the ϕ coordinate and show that the corresponding Euler-Lagrange equation of motion is given by (20pt)

$$\frac{d}{d\tau}\left[r^2\frac{d\phi}{d\tau}\right] = 0\,. \tag{10}$$

Varying the Lagrangian with respect to the ϕ coordinate:

$\frac{\partial L}{\partial \phi} = 0$ since L does not depend explicitly from ϕ

$$\frac{\partial L}{\partial \dot\phi} = \frac{1}{2}(L^2)^{-\frac{1}{2}}(-2r^2\dot\phi) = -\frac{1}{L}r^2\frac{d\phi}{d\sigma}$$

but $L\,d\sigma = d\tau \rightarrow \frac{1}{L}\frac{1}{d\sigma} = \frac{1}{d\tau}$

$$\frac{\partial L}{\partial \dot\phi} = -r^2\frac{d\phi}{d\tau}$$

Euler-Lagrangian equation:

$$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot\phi}\right) + \frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot\phi}\right) = 0$$

$$\frac{1}{L}\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot\phi}\right) = 0$$

$$\frac{1}{L}\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot\phi}\right) = 0$$

$$\left.\begin{aligned}\frac{1}{L}\frac{1}{d\sigma} &= \frac{1}{d\tau}\\ \frac{\partial L}{\partial \dot\phi} &= -r^2\frac{d\phi}{d\tau}\end{aligned}\right\} \longrightarrow \boxed{\frac{d}{d\tau}\left(r^2\frac{d\phi}{d\tau}\right) = 0}$$

c. Using this equation of motion read off the expressions for the Christoffel symbols $\Gamma^{\phi}_{\alpha\beta}$ for all α, β by comparing with the general expression of the geodesic equation (10pt)

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0\,. \tag{11}$$

the expression for the geodesic equation for $\alpha = \phi$ is

$$\frac{d^2\phi}{d\tau^2} + \Gamma^{\phi}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

that is:

$$\frac{d^2\phi}{d\tau^2} + \Gamma^{\phi}_{tt}\left(\frac{dt}{d\tau}\right)^2 + \Gamma^{\phi}_{rr}\left(\frac{dr}{d\tau}\right)^2 + \Gamma^{\phi}_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 + 2\,\Gamma^{\phi}_{r\phi}\frac{dr}{d\tau}\frac{d\phi}{d\tau} = 0$$

from b):

$$\frac{d}{d\tau}\left(r^2\frac{d\phi}{d\tau}\right) = 0 \rightarrow 2r\frac{dr}{d\tau}\frac{d\phi}{d\tau} + r^2\frac{d^2\phi}{d\tau^2} = 0$$

and

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} = 0$$ comparing the 2 expressions

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}$$

$$\Gamma^{\phi}_{tt} = \Gamma^{\phi}_{rr} = \Gamma^{\phi}_{\phi\phi} = 0$$

d. Calculate the expression for the Christoffel symbols $\Gamma^{\phi}_{\alpha\beta}$ also 'by hand' using the formula

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right) \tag{12}$$

and compare your result with the expression calculated in c. (20pt)

the general expression for the christoffel symbol is

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

with $\alpha = \phi$

from point a) :

$$g_{tt} = -\left(1 - \frac{2M}{r}\right), \quad g_{rr} = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}, \quad g_{\phi\phi} = r^2$$

all the other elements are zero

the only non zero derivative is $\frac{\partial g_{\phi\phi}}{\partial r} = 2r$ and the only $g_{\phi\delta} \neq 0$ is $g_{\phi\phi}$

$$g_{\phi\phi}\Gamma^{\phi}_{r\phi} = \frac{1}{2}\cdot 2r = r$$

$$\Gamma^{\phi}_{r\phi} = \frac{r}{g_{\phi\phi}} = \frac{r}{r^2} = \frac{1}{r} \rightarrow \boxed{\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}}$$

$$\Gamma^{\phi}_{tt} = \Gamma^{\phi}_{rr} = \Gamma^{\phi}_{\phi\phi} = 0$$

the result is the same as c)