Practice session: General Relativity

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Give motivations and/or derivations for your answers.

1. **A torus in two-dimensional Euclidean space**

Consider the metric

$$ds^2 = (b + a\sin\phi)^2 d\theta^2 + a^2 d\phi^2 \tag{1}$$

Which describes a torus in 2D Euclidean space in the spherical coordinate system (θ, ϕ), where a and b are the torus radius and the radius of its section respectively.

(a) Derive the equations of motion for this metric for coordinate θ, using the Lagrangian.

(b) Read of the non-zero Christoffel symbols from your equations of motion derived in (a), using the formula

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0. \tag{2}$$

(c) Also derive/compute all (consider both θ and ϕ) the non-zero Christoffel symbols 'by hand' using the formula

$$g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right). \tag{3}$$

2. **Geometry of de Sitter space**

Consider the metric,

$$\mathrm{d}s^2 = (1-r^2)\mathrm{d}t^2 - (1-r^2)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{28}$$

which is a solution of Einstein equation with positive cosmological constant (de Sitter spacetime).

(a) Consider a massive particle undergoing inertial motion in this metric and derive its equations of motion.

(b) From this point on, we will be mostly concerned with motion in the (t, r) coordinates, so in what follows you can assume that the coordinates θ, ϕ always remain fixed. Show that a particle initially placed at the origin (i.e. $r|_{t=0} = 0$) with zero velocity (i.e. with $\frac{\mathrm{d}r}{\mathrm{d}t}|_{t=0} = 0$ will always stay there.

(c) Show that particles away from $r = 0$ feel a force towards larger values of, r, and will thus move towards the surface $r = 1$. On the surface, $r = 1$ the metric has a (coordinate) singularity. This surface is called the de Sitter horizon.

(d) Write an expression for the proper distance from $r = 0$ to the de Sitter horizon, and show that it is finite.

(e) Consider a light ray emitted from $r = 0$ towards the de Sitter horizon. Calculate its orbit and show that in the (t, r) coordinates the light ray never crosses the horizon.

(f) As in the case of the Schwarzschild black hole, this is somewhat misleading. Define the analogue of Eddington Finkelstein coordinates $\bar{t}$, r, in which the outgoing light rays (i.e. those moving towards large values of r) move along straight lines $\bar{t} - r = \textit{constant}$.

(g) write the metric (28) in these new coordinates and show that it is smooth at $r = 1$ and can be extended past this surface to $r > 1$.

(h) Analyze the causal structure of the metric (28). This means, calculate the form of the in- and out-going lightcones, draw a spacetime diagram in the $\bar{t}$, r coordinates and plot qualitatively the form of the lightcones for ingoing (moving towards smaller r) and outgoing (moving towards larger r) lightrays.
From this diagram, argue that the surface $r = 1$ does indeed act like a horizon, that is, any object which starts in the region $r < 1$ and then crosses the surface $r = 1$ towards $r > 1$, will never be able to come back to the region $r < 1$. From the persepctive of an observer sitting at $r = 0$ this object is forever lost behind the de Sitter horizon.

3. **Expanding flat universe**

Consider the following metric describing an expanding flat Universe,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{72}$$

(a) Compute the Christoffel symbols in terms of $a(t)$.
(b) Derive the equations of motion for massive particles moving inertially in this spacetime
(c) Check that the orbits {x(t), y(t), z(t)} = constant, correspond to inertial motion.

From this point on, we focus on the specific case of an inflating Universe where,

$$a(t) = \exp\left(\sqrt{\frac{\Lambda}{3}}t\right) \tag{73}$$

(d) A light ray is emitted from the point {t, x, y, z} = {t_0 , 0, 0, 0} towards positive values of x. What is the orbit that the light ray will follow?
(e) Find the maximum value of the coordinate x that the light ray described above can reach.
(f) Using the metric compute the physical spacelike distance along the slice $t = t_0$ corresponding to the coordinate distance in x that you found in e).

4. **General to Special Relativity**

Assuming a general coordinate transformation, the components of vector fields **a** and **b** transform as follows:

$$a'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} a^{\beta}, \qquad b'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} b_{\beta}, \tag{118}$$

(a) Show that the components of a metric tensor transform under general coordinate transformations as:

$$g'_{\alpha\beta}(x) = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta}(x) \tag{119}$$

(b) Show that the scalar product between two vector fields:

$$\boldsymbol{v}(x) \cdot \boldsymbol{w}(x) = g_{\alpha\beta}(x) v^{\alpha}(x) w^{\beta}(x) \tag{120}$$

is invariant under the general coordinate transformations.

(c) Show that under a general coordinate transformation ($x'^{\alpha} = (\Lambda^{-1})^{\alpha}{}_{\beta} x^{\beta}$), where Λ^{-1} is a constant invertable matrix, the components of vector field **b** transform as:

$$b'_{\alpha} = \Lambda^{\beta}_{\alpha} b_{\beta} \tag{121}$$

(d) Now consider a local inertial frame at a Point P where $g_{\alpha\beta} = \eta_{\alpha\beta}$ and the first derivative of $g_{\alpha\beta}$ vanishes. Using eq. (119), show that the components of the Minkowski spacetime metric $\eta_{\alpha\beta}$ are invariant under general coordinate transformations (as defined in the previous subquestion). In other words, show that the conditions for which $\eta'_{\alpha\beta} = \eta_{\alpha\beta}$ are given by:

$$(\Lambda^T)^{\alpha}_{\gamma}(\Lambda)^{\gamma}_{\beta} = \mathbb{1}^{\alpha}_{\beta} \quad \text{where} \quad (\Lambda^T)^{\alpha}_{\beta} \equiv \eta^{\alpha\gamma}\eta_{\beta\delta}\Lambda^{\delta}_{\gamma} \tag{122}$$

5. **Einstein & Riemann**

(a) Show that Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{145}$$

can also be written as,

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} + \frac{T}{2-D} g_{\mu\nu} \right), \tag{146}$$

in which D is the dimension of space time.

(b) How many independent components does the Riemann tensor have in 2 dimensions?[1]

[1] Show explicitly that there are that amount of independent components.