Name: Student Number:

Mechanics and Relativity: M resit

February 13, 2025, Aletta Jacobshal
Duration: 120 mins

Before you start, read the following:

- There are 3 problems with subquestions, and you can earn 90 points in total. Your final grade is 1+(points)/10.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations:

$$\vec{F} = m\vec{a}\,, \quad K = \tfrac{1}{2}mv^2\,, \quad V = \tfrac{1}{2}kx^2\,, \quad I = \int x^2 dm\,, \quad \tau = I\alpha\,, \quad \vec{F}_{\text{centr}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})\,, \quad \vec{F}_{\text{Cor}} = -2m\vec{\omega} \times \vec{v}\,, \,,$$

and the Taylor expansions at small x:

$$(1 + ax)^b \approx 1 + abx + \ldots\,, \quad \sin(x) \approx x + \ldots \quad \cos(x) \approx 1 - \tfrac{1}{2}x^2 + \ldots\,. \tag{1}$$

Question 1: Coupled carts

Consider two carts of mass m that can move back and forth on an air rail, such that you can ignore friction in this question. The two carts are connected to each other with a spring with spring constant $3k$. Moreover, the left cart is attached to the left wall, and the right cart to the right wall, with springs with spring constants k.

(a) **(10 pts)** Write down Newton's second law for both carts. In other words, which second order differential equation governs the dynamics for x_1, and which one for x_2?

(b) **(10 pts)** One of the normal modes of this system has the carts moving out of phase (so always in opposite directions). Calculate the frequency of this normal mode.

(c) **(10 pts)** Now we add an external force that drives the system at a driving frequency ω_d, and we observe the amplitude of the oscillation of the left cart. Sketch what this amplitude roughly looks like as a function of ω_d. (You don't have to include information about the exact values of e.g. the frequency derived under b), only draw the qualitative features of this amplitude, such as peaks and/or behaviour at small and large ω_d.)

Question 2: Moment of inertia and linear acceleration

Consider a solid ball of mass M and radius R (with uniform mass density) that is placed on a slope that is tilted with respect to the ground with an angle θ (somewhere between 0 and 90 degrees).

(a) **(10 pts)** Calculate the moment of inertia of the ball with respect to its center of mass, using the fact that the moment of inertia of a hollow (instead of solid) ball of the same mass and radius is $I = \frac{2}{3}MR^2$. (The volume of a ball or radius R is $V = \frac{4}{3}\pi R^3$.)

(b) **(10 pts)** What is the linear acceleration of the center of mass as the ball rolls down the slope? For this subquestion, you can assume the no-slip condition. (If you did not find an answer at a), use $I = \frac{1}{6}MR^2$.)

(c) **(10 pts)** The fact that the ball is rolling down without slipping, instead of sliding down, is due to friction. How large should the coefficient μ of (static) friction at least be (as a function of the angle θ) such that the ball rolls down without slipping? (If you didn't find an answer at a) and/or b), use $I = \frac{1}{6}MR^2$ and $a = \frac{1}{4}Mg\sin(\theta)$ instead.)

Question 3: Turntable

Imagine a horizontal table that is rotating in a clockwise fashion as seen from above, with angular frequency ω. There is a billiard ball rolling on the table in such a manner that it has a fixed position in the (inertial) lab frame (so with respect to the ground and the room in which the turntable is located). The billiard ball has almost all its mass concentrated at its center, and hence you may neglect its moment of inertia (and hence the rotation of the ball around its center) in this question.

(a) **(15 pts)** Describe the motion of the billiard ball from the perspective of an observer who is rotating with the turntable, so the (non-inertial) turntable frame. Explain which fictitious forces play a role in this frame, what their magnitudes and directions are and how they combine to generate the observed motion in the turntable frame.

(b) **(15 pts)** Now imagine that the turntable (while rotating at constant ω) is not exactly horizontal, but has a small tilt (of angle θ) with respect to the ground. This results in an additional gravitational force on the billiard ball. Can one still place the billiard ball somewhere on the turntable, with some velocity with respect to the turntable, in order to achieve that it is constant in the lab frame? Explain your answer using the different forces; you are free to do so either from the lab frame or turntable frame, whichever would be simpler for you.