Name: Student Number:

# Mechanics and Relativity: M resit

February 13, 2025, Aletta Jacobshal
Duration: 120 mins

**Before you start, read the following:**

- There are 3 problems with subquestions, and you can earn 90 points in total. Your final grade is 1+(points)/10.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations:

$$\vec{F} = m\vec{a}\,, \quad K = \tfrac{1}{2}mv^2\,, \quad V = \tfrac{1}{2}kx^2\,, \quad I = \int x^2 dm\,, \quad \tau = I\alpha\,, \quad \vec{F}_{\text{centr}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})\,, \quad \vec{F}_{\text{Cor}} = -2m\vec{\omega} \times \vec{v}\,,\,,$$

and the Taylor expansions at small $x$:

$$(1+ax)^b \approx 1 + abx + \dots\,, \quad \sin(x) \approx x + \dots \quad \cos(x) \approx 1 - \tfrac{1}{2}x^2 + \dots\,. \tag{1}$$

## Question 1: Coupled carts

Consider two carts of mass $m$ that can move back and forth on an air rail, such that you can ignore friction in this question. The two carts are connected to each other with a spring with spring constant $3k$. Moreover, the left cart is attached to the left wall, and the right cart to the right wall, with springs with spring constants $k$.

(a) **(10 pts)** Write down Newton's second law for both carts. In other words, which second order differential equation governs the dynamics for $x_1$, and which one for $x_2$?

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - 3k(x_1 - x_2) = -4kx_1 + 3kx_2\,, \\ m\ddot{x}_2 &= -kx_2 - 3k(x_2 - x_1) = -4kx_2 + 3kx_1\,. \end{aligned} \tag{2}$$

*(5pts for correct outside forces proportional to k, 5pts for correct intermediate force proportional to 3k.)*

(b) **(10 pts)** One of the normal modes of this system has the carts moving out of phase (so always in opposite directions). Calculate the frequency of this normal mode.

This is the mode $x_1 - x_2$, with differential equation

$$m(\ddot{x}_1 - \ddot{x}_2) = -7k(x_1 - x_2)\,. \tag{3}$$

It therefore oscillates with 'angular' frequency $\omega^2 = 7k/m$ and the frequency $f = \omega/(2\pi)$.
*(5pts for correct identification of $x_1 - x_2$ and its diff eq, 5pts for correct derivation of frequency from it.)*

(c) **(10 pts)** Now we add an external force that drives the system at a driving frequency $\omega_d$, and we observe the amplitude of the oscillation of the left cart. Sketch what this amplitude roughly looks like as a function of $\omega_d$. (You don't have to include information about the exact values of e.g. the frequency derived under b), only draw the qualitative features of this amplitude, such as peaks and/or behaviour at small and large $\omega_d$.)

It starts at some finite value, has two peaks (that shoot up to infinity in the absence of friction), and then decreases to zero at large driving frequency.
*(5pts for two peaks (either finite or infinite), 5pts for correct asymptotic behaviours.)*

## Question 2: Moment of inertia and linear acceleration

Consider a solid ball of mass $M$ and radius $R$ (with uniform mass density) that is placed on a slope that is tilted with respect to the ground with an angle $\theta$ (somewhere between 0 and 90 degrees).

(a) **(10 pts)** Calculate the moment of inertia of the ball with respect to its center of mass, using the fact that the moment of inertia of a hollow (instead of solid) ball of the same mass and radius is $I = \frac{2}{3}MR^2$. (The volume of a ball or radius $R$ is $V = \frac{4}{3}\pi R^3$.)

We want to integrate over shells from 0 to $R$. These have a mass $dm = \rho 4\pi r^2 dr$ with $\rho$ the density. Their contribution to the moment of inertia is $dI = \frac{2}{3}dmr^2 = \frac{8}{3}\pi\rho r^4 dr$. Then the full moment of inertia of the solid ball is

$$I = \int_0^R dI = \tfrac{8}{3}\pi\rho\tfrac{1}{5}r^5 = \tfrac{2}{5}MR^2\,, \tag{4}$$

where we used $M = \rho V$ in the last step.
*(5pts for correct set-up of integral over successive shells and correct infinitesimal moment of inertia $dI$, 5pts for correct integral into full $I$.)*

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(b) **(10 pts)** What is the linear acceleration of the center of mass as the ball rolls down the slope? For this subquestion, you can assume the no-slip condition. (If you did not find an answer at a), use $I = \frac{1}{6}MR^2$.)

Decomposing the forces along the two directions (along the slope, ortogonal to the slope) leads to

$$N = Mg\cos(\theta)\,, \qquad Ma = Mg\sin(\theta) - F_{\text{friction}}\,. \tag{5}$$

The torque wrt the CM is

$$\tau = I\alpha = RF_{\text{friction}}\,. \tag{6}$$

Angular and linear acceleration are related via $a = R\alpha$. Using $I = \frac{2}{5}MR^2$, this can be solved to give $a = \frac{5}{7}g\sin(\theta)$.
(There is an alternative way of deriving this result via energy conservation (kinetic, rotational and potential), which of course is allowed as well.)
*(5pts for correct forces and torque, 5pts for correct derivation of $a$ from these.)*

(c) **(10 pts)** The fact that the ball is rolling down without slipping, instead of sliding down, is due to friction. How large should the coefficient $\mu$ of (static) friction at least be (as a function of the angle $\theta$) such that the ball rolls down without slipping? (If you didn't find an answer at a) and/or b), use $I = \frac{1}{6}MR^2$ and $a = \frac{1}{4}Mg\sin(\theta)$ instead.)

The magnitude of the friction force is at most $\mu N$. Using the expressions for the friction and normal forces above, this implies $Mg\sin(\theta) - Ma \leq \mu Mg\cos(\theta)$. With the derived value of $a$, this leads to $\mu \geq \frac{2}{7}\tan(\theta)$.
*(5pts for bound on friction force, 5pts for subsequent analysis.)*

## Question 3: Turntable

Imagine a horizontal table that is rotating in a clockwise fashion as seen from above, with angular frequency $\omega$. There is a billiard ball rolling on the table in such a manner that it has a fixed position in the (inertial) lab frame (so with respect to the ground and the room in which the turntable is located). The billiard ball has almost all its mass concentrated at its center, and hence you may neglect its moment of inertia (and hence the rotation of the ball around its center) in this question.

(a) **(15 pts)** Describe the motion of the billiard ball from the perspective of an observer who is rotating with the turntable, so the (non-inertial) turntable frame. Explain which fictitious forces play a role in this frame, what their magnitudes and directions are and how they combine to generate the observed motion in the turntable frame.

From the turntable perspective, the ball makes circular motion (or radius $R$ in a counterclockwise direction. It does so under the influence of two fictitious forces: the Coriolis force (pointing inwards) and the centrifugal force (pointing outwards). Their magnitudes are$2m\omega^2 R$ (using $v = \omega R$) and $m\omega^2 R$, resp. The resulting force is therefore $m\omega^2 R$ pointing inwards, which is exactly the force needed to make a circle (all in the turntable frame).

*(5pts for correct description of motion, 5pts for correct fictitious forces (direction and magnitude), and 5pts for combination into net force.)*

(b) **(15 pts)** Now imagine that the turntable (while rotating at constant $\omega$) is not exactly horizontal, but has a small tilt (of angle $\theta$) with respect to the ground. This results in an additional gravitational force on the billiard ball. Can one still place the billiard ball somewhere on the turntable, with some velocity with respect to the turntable, in order to achieve that it is constant in the lab frame? Explain your answer using the different forces; you are free to do so either from the lab frame or turntable frame, whichever would be simpler for you.

No, this is not possible.

In the lab frame, this is simplest to see since there are no fictitious forces, only gravity - so there are no cancellations and the ball cannot help but rolling off.

In the turntable frame, this is complicated to see since on top of the fictitious forces in the previous subquestion, there is now also gravity, whose direction depends on time.

(A final possibility is to assume that the ball is constant in the lab frame, and approximate the part of the turntable that it samples as an infinite, straight strip of turntable material over which it moves at constant velocity. Again, this is an inertial frame, so no fictitious forces, only gravity - and hence the ball will accelerate, violating the initial assumption of constant velocity.)

*(5pts for correct answer, 10pts for correct reasoning.)*