Mechanics and Relativity: M1

Mock exam

Duration: 60 mins

Before you start, read the following:

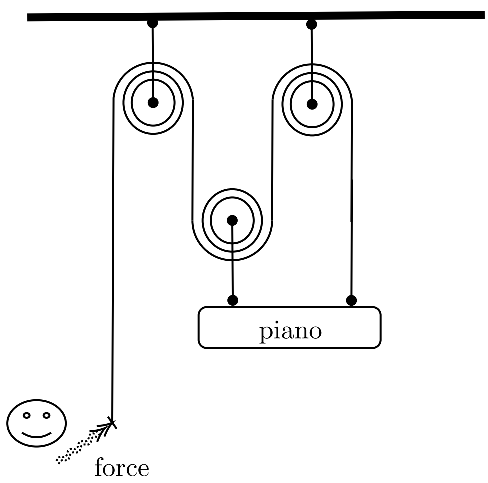
- There are 2 problems with subquestions, and you can earn 90 points in total (45 per problem). Your final grade is 1+(points)/10.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations and values:

$$F = ma\,, \quad E = mc^2\,, K = \tfrac{1}{2}mv^2\,, \quad V = mgh\,, \quad V = -\tfrac{1}{2}kx^2\,, \quad g \approx 10 m/s^2\,. \tag{1}$$

Question 1: Pulling up a piano

Consider an extended Atwood machine as indicated in the picture, with a single rope going through three pulleys and connected to a heavy piano of mass M. You can assume the rope is massless and does not stretch, and the pulleys are massless and have no friction. Moreover, for some reason the piano cannot rotate or tilt, and will only move up or down.



(a) **(10 pts)** What force does the person need to exert to achieve a static configuration?

The piano is pulled upwards by three times the tension in the rope, and hence the person needs to exert $F = 1/3Mg$.

(b) **(10 pts)** When instead one wants to accelerate the piano upwards with 1 m/s^2, by which percentage does the person have to increase the force exerted (as compared to the static configuration)? Briefly explain your answer.

Instead of overcoming g the person now needs to overcome $g + a$. So 10 % extra force for 1 m/s^2.

(c) **(10 pts)** What is the minimal amount of work that the person has to perform while pulling the piano up over a distance L? Indicate how this follows from the force exerted and the length of rope pulled in.

Minimal amount of work: just above the force $1/3Mg$ (so the piano arrives with minimal kinetic energy). Exerted over a length of $3L$ (this is what the person has to pull in). So work done amounts to MgL. (This is of course equal to the potential energy gain of the piano.)

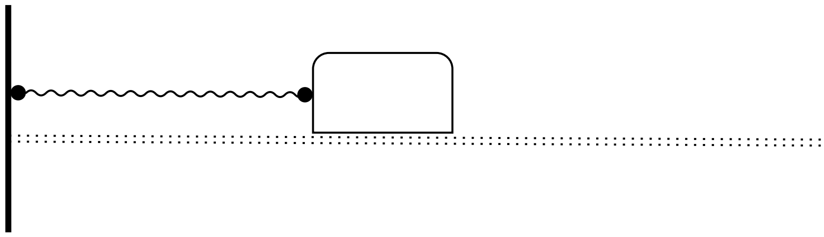
(d) **(15 pts)** Suppose the lower pulley would block completely, i.e. would have infinite friction (the rope would be stuck there). How does this affect your calculation at (c)? In particular, how does it affect the force exerted, length of rope pulled in and the total amount of work?

In this case the force would be Mg (i.e. tripled) and the length would be L (factor three smaller), so again total work MgL (the same). Same potential energy gain.

Name: Student Number:

Question 2: Oscillations of a cart

Consider a single cart of mass m on a rail, connected on the left side to a spring with spring constant k, see the picture.



(a) **(15 pts)** Include a damping force equal to $-2m\gamma\dot{x}$ with a constant γ. From Newton's laws governing the dynamics, what is the motion of the cart $x(t)$ in the overdamped case? Which timescales govern the dynamics in this case?

Newton's eq with damping and spring force, and its solution in overdamped case ($\gamma^2 > k/m$):

$$\ddot{x} + 2\gamma\dot{x} + k/m = 0\,, \quad x = e^{-\gamma t}(Ae^{\sqrt{\gamma^2 - k/m}t} + Be^{-\sqrt{\gamma^2 - k/m}t})\,. \tag{2}$$

Both are exponential damping time dependences, with timescales given by the inverse of the coefficients in the argument of the exponentials:

$$\frac{1}{\gamma \pm \sqrt{\gamma^2 - k/m}}\,. \tag{3}$$

(b) **(15 pts)** Suppose we now switch off damping and instead turn on a driving force, for instance as the left wall starts moving periodically, of the form $F\sin(\omega_d t)$. Derive the particular (or inhomogeneous) solution in this case.

We have to solve the differential equation

$$\ddot{x} + kx/m = F\sin(\omega_d t)/m\,. \tag{4}$$

First of all, we rewrite the sine function into a sum of two driving forces $exp(\pm i\omega_d t)$, and then derive the corresponding particular solution. It is given by

$$x = \frac{F/m}{k/m - \omega_d^2}\frac{-i}{2}(e^{i\omega_d t} - e^{-i\omega_d t})\,, \tag{5}$$

which can be rewritten as

$$x = \frac{F/m}{k/m - \omega_d^2}\sin(\omega_d t) \tag{6}$$

(c) **(15pts)** When including both damping and driving, the amplitude of the particular solution peaks at a driving frequency given by $\omega_d^2 = k/m - 2\gamma^2$. Does it make sense to talk about resonance in the critically damped case? At which driving frequency is the amplitude largest?

No, critically damped corresponds to $\gamma^2 = k/m$ and hence there is no peak corresponding to a resonance for some specific driving frequency. Instead, the amplitude is largest at ω_d zero.