Name: Student Number:

Mechanics and Relativity: M2

Mock exam
Duration: 90 mins

Before you start, read the following:

- There are 3 problems with subquestions, and you can earn 90 points in total. Your final grade is 1+(points)/10.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations and values:

$$F = ma\,, \quad \vec{L} = \vec{r}\times\vec{p}\,, \quad T_{\rm kin} = \tfrac{1}{2}m\vec{v}\cdot\vec{v}\,, \quad T_{\rm rot} = \tfrac{1}{2}I_z\omega^2\,, \quad \tau = I\alpha\,, \quad T^2 = \frac{4\pi^2 a^3}{GM_{\rm Sun}}\,. \tag{1}$$

Question 1: Conservative forces

(a) **(10 pts)** What is the condition on a force $\vec{F}(\vec{x})$ in three dimensions to be conservative?

It has to be curl-free, that is $\vec{\nabla}\times\vec{F} = 0$.

(b) **(10 pts)** Conservative forces can be written in terms of the gradient of a potential energy $V(\vec{x})$. Show that the sum of kinetic and potential energy of a particle moving in three dimensions is time-independent.

$$\frac{d}{dr}(\tfrac{1}{2}m\vec{v}\cdot\vec{v} + V) = m\vec{v}\cdot\vec{a} + \vec{\nabla}V\cdot\vec{v} = (m\vec{a} - \vec{F})\cdot\vec{v} = 0\,. \tag{2}$$

(c) **(10 pts)** Consider a star whose density varies as a function of the distance to its core: it is given by $\rho = \sqrt{1-(r/r_0)^2}\rho_0$, where r_0 is the outer surface of the star and ρ_0 is the density at the core. Is the resulting gravitational force outside of the star a conservative force? Briefly explain your answer.

Yes it is conservative. You can see this in different ways. One is that the gravitational field of any spherically symmetric mass distribution outside the object is the same as that of a point source. Alternatively, you can think of this star as made up of many shells, which are all conservative, and the sum of conservative forces is still conservative.

Question 2: Angular momentum

(a) **(15 pts)** Calculate the moment of inertia of a solid sphere of mass M and radius R. Assume the density to be constant. In your derivation, you can use the fact that a hollow sphere of the radius r and mass m (with all mass at the outer shell) has $I = \frac{2}{3}mr^2$.

See derivation in the book, $I = \frac{2}{5}MR^2$.

(b) **(10 pts)** Two solid objects of equal mass are rolling down a plane, both of constant density: a cylinder and a sphere. Which one arrives first at the bottom of the plane? Briefly explain your answer. (You should be able to argue this on physical grounds, independent of your answer at (a)).

The sphere will beat the cylinder, for the following reason. In this set-up, rotation will slow down an object, so the important question is which object has a larger moment of inertia. This will be the cylinder, as it has a larger fraction of its mass away from the center, as compared to the sphere. Therefore the cylinder will be more reluctant to move, and hence slower.

Question 3: Rotating sticks

Consider a stick of fixed length $L = 1$ that extends from $x = 0$ to $x = 1$. It has a total mass $M = 1$ but varying density $\rho(x) = a + 2(1-a)x$, that can be tuned by changing the parameter a (which takes values from 0 to 2).

(a) **(10 pts)** At which location (i.e. at which x-value) is the center of mass of this stick (as a function of the parameter a)?

$$x_{\rm CM} = \int x\rho(x)dx = (\tfrac{1}{2}ax^2 + \tfrac{2}{3}(1-a)x^3)|_{x=0}^{x=1} = \tfrac{2}{3} - \tfrac{1}{6}a\,. \tag{3}$$

(b) **(10 pts)** What is the moment of inertia of this stick around its $x = 0$ point (as a function of the parameter a)?

$$I = \int x^2\rho(x)dx = (\tfrac{1}{3}ax^3 + \tfrac{1}{2}(1-a)x^4)|_{x=0}^{x=1} = \tfrac{1}{2} - \tfrac{1}{6}a\,. \tag{4}$$

(c) **(15 pts)** Now attach the $x = 0$ point of the stick to a vertical wall, and allow it to rotate around this point. Starting from a horizontal position, what is the initial angular acceleration α (as a function of the parameter a)? For which of the following three values will it accelerate fastest: $a = 0, 1, 2$? (If you did not find an answer at the previous two questions, use $x_{\rm CM} = 3 - a$ and $I = 2 - a$ instead.)

$$\tau = I\alpha \quad \Rightarrow \quad gx_{\rm CM} = I\alpha\,, \quad \Rightarrow \quad \alpha = g\frac{\frac{2}{3} - \frac{1}{6}a}{\frac{1}{2} - \frac{1}{6}a} = g\frac{a-4}{a-3}\,. \tag{5}$$

The angular acceleration is therefore largest for the case $a = 2$.