

Mechanics and Relativity: M2

December 17, 2024, Aletta Jacobshal

Duration: 90 mins

Before you start, read the following:

- There are 3 problems with subquestions, and you can earn 90 points in total. Your final grade is $1 + (\text{points})/10$.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations and values:

$$F = ma, \quad \vec{L} = \vec{r} \times \vec{p}, \quad T = \frac{1}{2} I_z \omega^2, \quad \tau = I_z \alpha. \quad (1)$$

Question 1: Central and conservative forces

- (a) **(10 pts)** When a force is conservative, which physical quantity/ies are conserved? And which ones when it is central? (No explanations needed.)

Conservative: total energy. Central: total energy and angular momentum.

(5pts for correct answer for Conservative, 5pts for correct answer for Central.)

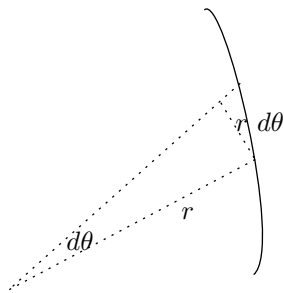
- (b) **(10 pts)** Consider a satellite in the gravitational force field of the Earth and the Moon (and neglect the rest of the Universe). Is this gravitational force conservative? Is it central? (No explanations needed.)

Conservative: yes. Central: no.

(5pts per correct yes/no answer.)

Name:

Student Number:



- (c) **(10 pts)** Prove Kepler's second law which states that the area swept out by the Earth as it orbits the Sun is the same for equal infinitesimal time intervals (see figure). In your proof, you can use the conservation of angular momentum.

The area swept out per infinitesimal time element dt is given by (from triangle in the picture) $A = \frac{1}{2}r \cdot r d\theta/dt$. This can be related to the angular momentum of a particle, which is given by $L = mr^2\dot{\theta}$. Therefore we have $A = \frac{1}{2m}L$, and hence the fact that angular momentum is conserved implies that $dA/dt = 0$.

(5pts for correct identification of area of triangle, and 5pts for correct computation.)

Question 2: Moment of inertia

- (a) **(10 pts)** Calculate the moment of inertia of a stick of length D and mass m , where the axis of rotation is orthogonal to the stick and goes through its center of mass.

$$I = \int_{-L/2}^{L/2} \rho x^2 dx = \frac{1}{3}x^3 \Big|_{-L/2}^{L/2} = \frac{1}{12}\rho L^3 = \frac{1}{12}ML^2. \quad (2)$$

(5pts for correct integrand $\rho x^2 dx = x^2 dm$, 5pts for correct evaluation of integral)

Name:

Student Number:

- (b) **(10 pts)** The moment of inertia of a massive ring of total mass M and radius R , where all mass is located at the rim and where the axis of rotation is orthogonal to the plane of the ring and goes through its center of mass, is given by MR^2 . Calculate the moment of inertia around the same axis of rotation when a wheel is built by adding two spokes to this ring (thus making roughly this figure: Θ). The total mass will be $4M/3$ (of which M resides in the outer ring and $M/3$ in the spokes). (If you didn't get an answer at (a), use $\frac{5}{7}mD^2$.)

Moment of inertia of ring: MR^2 .

Moment of inertia of spokes: $\frac{1}{12}mL^2 = \frac{1}{12}(M/3)(2R)^2 = \frac{1}{9}MR^2$.

So the combination has $\frac{10}{9}MR^2$.

(5pts for correct moment of inertia of spokes, 5pts for addition of the two)

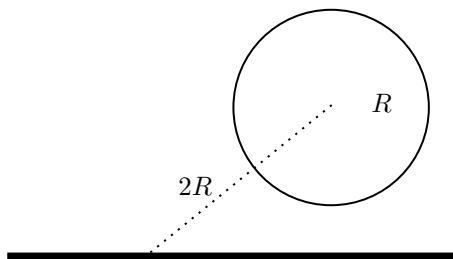
- (c) **(10 pts)** A Formula 1 maniac adds an additional eight spokes to the wheel (of the same material as before, thus doubling the total weight of the wheel). When rolling down from a hill (where you can ignore air resistance etc), which wheel will be faster? Assume that the wheels do not slip. Briefly explain your answer, e.g. in two sentences. (You should be able to argue this on physical grounds, independent of your answer at (b). No calculations necessary).

The wheel with many spokes will be faster, as it will have relatively more mass located to the axis of rotation, and hence will have a smaller moment of inertia (relative to its mass). It will therefore be slowed down less, compared to the wheel with only two spokes.

(5pts for answering that the one with many spokes will be faster, 5pts for reasoning in terms of comparison of moment of inertia)

Question 3: Falling backwards - or not...

Consider a very skinny person with a massive bike wheel; all of the mass M is concentrated in (the rim of) the wheel with radius R , while the length of the person is $2R$. The person is leaning backwards at an angle of 45 degrees, without initial velocity. The feet of the person cannot slip on the ground, and hence remain at the same location.



Name: _____

Student Number: _____

- (a) **(10 pts)** What is the torque exerted by gravity on this configuration?

The arm of the lever is $2R$, and the gravitational force is Mg . There is a 45 degree angle between them. The torque is therefore $\sqrt{2}MgR$.

(10pts for correct answer; partial 5pts for correct lever arm / orthogonal part of gravitational force.)

- (b) **(10 pts)** In the case where the wheel does not turn by itself (in other words, the person and wheel together behave as a single rigid body), what is the (initial) angular acceleration (in terms of e.g. M, g, R etc) of the person?

The moment of inertia of the entire object is: around the CM it's MR^2 , and the point of rotation is $2R$ away. Therefore, with the parallel axis theorem, we have $I = M(2R)^2 + MR^2 = 5MR^2$. The angular acceleration is given by $\alpha = \tau/I = \sqrt{2}g/(5R)$.

(5pts for correct total moment of inertia, 5pts for correct application of torque - angular acceleration relation.)

- (c) **(10 pts)** By continuously speeding up the wheel separately, the person can hover indefinitely at this diagonal position. What would the angular acceleration (in terms of e.g. M, g, R etc) of the wheel have to be to achieve this?

The torque will induce an increase in angular momentum. When the person does not fall, but instead the wheel starts and accelerates spinning, its angular momentum is given by $L = MR^2\omega$. Its increase $dL/dt = mR^2\alpha$ (where ω and α are the angular velocity and acceleration of the wheel). So this means that $\alpha = \tau/I = \sqrt{2}g/R$.

(10pts for correct answer - partial: 5pts for correct angular momentum or its time derivative in terms of angular acceleration.)