Name: Student Number:

Mechanics and Relativity: M3

January 22, 2024, Aletta Jacobshal
Duration: 120 mins

Before you start, read the following:

- There are 3 problems with subquestions, and you can earn 90 points in total. Your final grade is 1+(points)/10.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations:

$$\vec{F} = m\vec{a}\,, \quad \vec{L} = \vec{r}\times\vec{p}\,, \quad \vec{\tau} = \vec{r}\times\vec{F}\,, \quad \vec{F}_{\text{centr}} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r})\,, \quad \vec{F}_{\text{Cor}} = -2m\vec{\omega}\times\vec{v}\,,$$

and the trigonometric identities $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$ and $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$.

Question 1: Principal axes

(a) **(10 pts)** Consider a ring standing up vertically, with mass M and radius R, which is so flat that it's effectively a planar object. Indicate the three principal axes for rotation around the lowest point of the ring in one or several clear drawings.

For full points you should indicate the following in your drawings: using the same coordinate system as drawn later on at question 2, the principal axes are x, y, z with the origin located at the pivot point (lowest point of the ring) and the ring in the (x, z) plane.
(10pts for all axes correct, partial: 5pts for one of the two axes in plane of ring correct.)

(b) **(10 pts)** Of the three principal moments, associated with the object and principal axes of the previous subquestion, calculate the one that has the *smallest* value.

The smallest one will associated to the axes that crosses the ring in two places. Its principal moment is given by

$$I = \int r^2 dm = \int \rho R^3 sin^2(\theta) d\theta = \rho\pi R^3 = \tfrac{1}{2} M R^2 \,, \tag{1}$$

using the identity $sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$ given above.
(5pts for identifiation of correct axis, 5pts for correct calculation.)

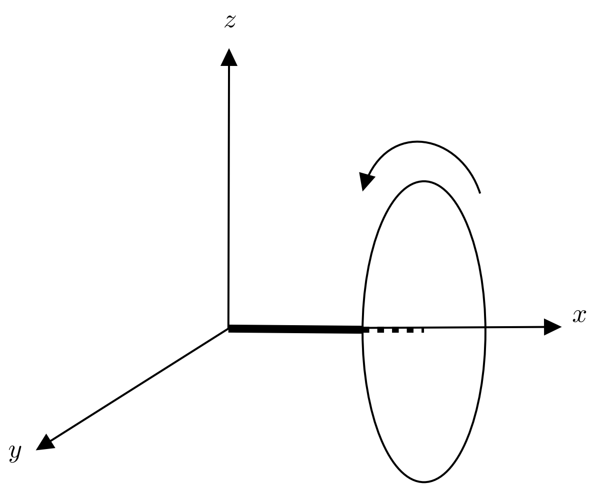
(c) **(10 pts)** Now instead consider rotation around the center of mass of the ring. Will all three principal moments have different values, or will some be equal? Briefly explain your answer, e.g. in one or two sentences. (No calculations needed.)

The principal moments associated with the two axes in the plane of the ring will be equal, due to the rotational symmetry of the ring around the orthogonal axis, the third principal axis.
(5pts for correct answer, 5pts for correct argument (symmetry))

Question 2: Suspended bike wheel with spin and precession

Consider a bike wheel that is suspended in mid air on one of its sides, allowing it to spin and precess. Take all mass M to be at the rim of the wheel at radius R, such that its moment of inertia around the axis of the wheel is MR^2. Take the distance from the center of mass to the suspension point to be x_0. Finally, take the spinning direction to be clockwise as seen from the suspension point with angular frequency ω_3. You can neglect friction.

(a) **(15 pts)** Using the torque and rate of change of angular momentum, derive the angular precession frequency Ω of the bike wheel, and its direction.



The torque due to gravity has magnitude $\tau = Mgx_0$ and points into the paper. The angular momentum has magnitude $MR^2\omega_3$ and points to the right. From these, the precession frequency is

$$\Omega = \frac{Mgx_0}{MR^2\omega_3}\,. \tag{2}$$

The precession direction is counterclockwise, as seen from above.
(10pts for correct precession frequency, 5pts for correct precession direction; partial: in total 5pts for having both torque and angular momentum correct

(b) **(15 pts)** The total angular momentum $\vec{L}$ of any body is always composed of the angular momentum *of* the center of mass, $\vec{R} \times \vec{P}$, and the angular momentum *around* the center of mass, $\vec{L}_{\mathrm{CM}}$. In the above situation, what are the magnitudes of both components of angular momentum? Also, clearly indicate their directions in a drawing.

Using that the linear speed due to rotation is $V = x_0\Omega$, the component $\vec{R} \times \vec{P}$ has magnitude $x_0^2 M\Omega$ and is pointing upwards. The angular momentum has magnitude $MR^2\omega_3$ and points to the right.
(5pts per correct magnitude (so 10pts in total), and 5pts for having both directions correct.)

Question 3: Fictitious forces
Imagine you live somewhere on Earth at a latitude of 45 degrees, so exactly in between the North Pole and the equator. Take the Earth to be a perfect sphere with a radius of 6400 km and an angular speed of $\omega = 7 \cdot 10^{-5}$ 1/s. Consider a cloud above your house moving westwards with speed v, and restrict its motion to the surface of the Earth (i.e. no vertical motion).

(a) **(10 pts)** Out of the three fictitious forces given by translation, centrifugal and Coriolis,

$$\vec{F}_{\text{fict}} = -m\ddot{\vec{R}}_S - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}\,, \tag{3}$$

which one(s) will act on this (otherwise free) cloud? Calculate their magnitude(s) (in terms of v and ω) and their direction(s), in the plane spanned by the surface of the Earth.

The centrifugal force is pointing South, and has a magnitude $\frac{1}{2}m\omega^2 R$ (where the numerical factor gets a $1/\sqrt{2}$ from the latitude and another one from the projection onto the surface). The Coriolis force is pointing North, and has a magnitude $\sqrt{2}m\omega v$ (with one factor of $\sqrt{2}$ reduction due to projection).
(5ts for correct Coriolis force (magnitude and direction), 5pts for correct centrifugal force (magnitude and direction).

(b) **(10 pts)** For this cloud to trace out a circle around the Earth at constant latitude, it needs to be subject to a net force pointing north with magnitude $\frac{1}{2}m\omega^2 R$ (acting as centripetal force). Can the fictitious forces give rise to such a net force, what condition(s) need to be met for this (for instance in terms of the speed v)?

Yes, this requires the Coriolis force to be double the centrifigal force, such that it is converted into a centripetal effect. This happens when $v = \omega R/\sqrt{2}$.
(10pts for correct answer)

(c) **(10 pts)** Now consider the Earth from the perspective of an inertial observer located in outer space. Imagine that the center of mass of the Earth is at a constant location in this frame (and thus neglect the effect of the Moon, Sun etc.). What kind of motion does the cloud display as measured by this observer?

For the speed as derived under b): in the inertial frame of the outer space observer, the Earth is rotating (all points moving to the East), and the cloud is standing still in this inertial frame. When going faster: it moves to the West (overtaking the rotation of the Earth) and is curving towards the North. When going slower: it moves to East (carried with the rotation of the Earth) and is curving towards the South.
(10pts for correct answer)