Mathematical Physics 2025
Mock Exam

Instructions:

- Write your student number on each sheet you submit.
- You are allowed to bring one A4 cheat sheet (double-sided).
- No calculators, textbooks, or digital devices are allowed.
- Write clearly and legibly. Show all necessary steps in your calculations and clearly state any assumptions or theorems used.
- If you use a convention that is not defined in the lectures or textbooks, you must explain it clearly. Otherwise, points will be deducted.
- There are four problems in total. The total score is 100 points. This exam counts for 70% of your final grade.

Useful Identities and Equations

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sin^2 a + \cos^2 a = 1$$

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\, dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}, \quad \text{for } a > 0$$
$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \frac{d^2 u}{dx^2} = 0 \quad \text{or} \quad \nabla^2 u = 0 \quad \text{(in higher dimensions)}$$

Problem 1: (20pts)
Given the function

$$f(x) = \frac{x^3}{3 - x^2}$$

(a) Derive the power series representation of $f(x)$ centered at $x = 0$.

(b) Determine the radius of convergence of the series and the interval of convergence.

Problem 2: (20pts)
Consider the following differential equation:

$$x^2y'' + 5xy' + (4x^2 + 3)y = 0$$

Solve the differential equation about the singular point $x_0 = 0$ following these steps:

(a) Obtain the roots of the indicial equation. Do they admit two linearly independent solutions? Show your work clearly and justify your steps.

(b) Find the first 6 coefficients of the series in terms of a_0.

Problem 3: (25pts)
Consider solving the Heat Equation using Fourier series for a metal rod of length L.

(a) Give the boundary conditions that lead to energy dissipation in the rod.

(b) Give the boundary conditions that ensures the conservation of energy in the rod.

(c) Given initial condition $u(x, 0) = f(x)$ where $u(x, 0)$ is found as

$$u(x,0) = \sum_0^\infty A_n \cos(\frac{n\pi x}{L}), \qquad \text{and} \quad f(x) = \begin{cases} 1, & 0 < x < L/2 \\ -1, & L/2 < x < L \end{cases}$$

Obtain the full solution.

Problem 4: (25pts)

(a) The Fourier transform of an absolutely integrable function $f(x)$. Show that

$$\mathcal{F}[xf(x)] = p\frac{d}{dw}\left\{\mathcal{F}[f(x)]\right\}.$$

where p is a constant. Find the value of p.

(b) The Airy function $\mathrm{Ai}(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - xy = 0.$$

Use Fourier transforms to show that the Airy function $\mathrm{Ai}(x)$ can be expressed as

$$\mathrm{Ai}(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \cos\left(Aw^3 + Bxw\right)\, dw,$$

where A, and B are constants.

Determine the values of the constants A and B. Show your work.

Hint: You may use the result of the previous subquestion (4.a). If you can't find p you may express A and B in terms of p.