Mathematical Physics 2025
Mock Exam

Instructions:

- Write your student number on each sheet you submit.
- You are allowed to bring one A4 cheat sheet (double-sided).
- No calculators, textbooks, or digital devices are allowed.
- Write clearly and legibly. Show all necessary steps in your calculations and clearly state any assumptions or theorems used.
- If you use a convention that is not defined in the lectures or textbooks, you must explain it clearly. Otherwise, points will be deducted.
- There are four problems in total. The total score is 100 points. This exam counts for 70% of your final grade.

Useful Identities and Equations

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sin^2 a + \cos^2 a = 1$$

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\, dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}, \quad \text{for } a > 0$$
$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \frac{d^2 u}{dx^2} = 0 \quad \text{or} \quad \nabla^2 u = 0 \quad \text{(in higher dimensions)}$$

Problem 1: (20pts)
Given the function

$$f(x) = \frac{x^3}{3 - x^2}$$

(a) Derive the power series representation of $f(x)$ centered at $x = 0$.

(b) Determine the radius of convergence of the series and the interval of convergence.

Problem 2: (20pts)
Consider the following differential equation:

$$x^2 y'' + 5xy' + (4x^2 + 3)y = 0$$

Solve the differential equation about the singular point $x_0 = 0$ following these steps:

(a) Obtain the roots of the indicial equation. Do they admit two linearly independent solutions? Show your work clearly and justify your steps.

(b) Find the first 6 coefficients of the series in terms of a_0.

Problem 3: (25pts)
Consider solving the Heat Equation using Fourier series for a metal rod of length L.

(a) Give the boundary conditions that lead to energy dissipation in the rod.

(b) Give the boundary conditions that ensures the conservation of energy in the rod.

(c) Given initial condition $u(x, 0) = f(x)$ where $u(x, 0)$ is found as

$$u(x, 0) = \sum_{n=0}^{\infty} A_n \cos(\frac{n\pi x}{L}), \qquad \text{and} \quad f(x) = \begin{cases} 1, & 0 < x < L/2 \\ -1, & L/2 < x < L \end{cases}$$

Obtain the full solution.

Problem 4: (25pts)

(a) The Fourier transform of an absolutely integrable function $f(x)$. Show that

$$\mathcal{F}[xf(x)] = p\frac{d}{dw}\left\{\mathcal{F}[f(x)]\right\}.$$

where p is a constant. Find the value of p.

(b) The Airy function $\mathrm{Ai}(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - xy = 0.$$

Use Fourier transforms to show that the Airy function $\mathrm{Ai}(x)$ can be expressed as

$$\mathrm{Ai}(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \cos\left(Aw^3 + Bxw\right)\, dw,$$

where A, and B are constants.

Determine the values of the constants A and B. Show your work.

Hint: You may use the result of the previous subquestion (4.a). If you can't find p you may express A and B in terms of p.

Problem 1:

(a) Power Series Representation:

We aim to express $f(x)$ as a power series centered at $x = 0$. Start by rewriting the denominator to match the form of a geometric series:

$$\frac{1}{3 - x^2} = \frac{1}{3} \cdot \frac{1}{1 - \frac{x^2}{3}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x^2}{3} \right)^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{3^{n+1}}$$

Now multiply the series by x^3:

$$f(x) = x^3 \cdot \frac{1}{3 - x^2} = x^3 \sum_{n=0}^{\infty} \frac{x^{2n}}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{x^{2n+3}}{3^{n+1}}$$

Final power series:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+3}}{3^{n+1}}}$$

(b) Radius and Interval of Convergence:

The power series is derived from the geometric series

$$\frac{1}{1 - \frac{x^2}{3}} = \sum_{n=0}^{\infty} \left(\frac{x^2}{3} \right)^n$$

which converges when:

$$\left| \frac{x^2}{3} \right| < 1 \quad \Rightarrow \quad x^2 < 3 \quad \Rightarrow \quad |x| < \sqrt{3}$$

So, the radius of convergence is:

$$\boxed{R = \sqrt{3}}$$

Since the original function $f(x) = \frac{x^3}{3-x^2}$ is undefined at $x = \pm\sqrt{3}$ (vertical asymptotes), the series diverges at the endpoints.

Therefore, the interval of convergence is:

$$\boxed{(-\sqrt{3}, \sqrt{3})}$$

Problem 2: We solve about the singular point $x_0 = 0$ using Frobenius.

(a) Assume $y = x^r \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \\ y'' &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \end{aligned}$$

Sub into the ODE:

$$x^2 y'' + 5xy' + (4x^2 + 3)y = 0$$

First collect powers of x^{n+r}:

$$\sum_{n=0}^{\infty} a_n \left[(n+r)^2 + 4(n+r) + 3\right] x^{n+r} + \sum_{n=0}^{\infty} 4a_{n-2} x^{n+r} = 0$$

So recurrence:

$$a_n \left[(n+r)^2 + 4(n+r) + 3\right] + 4a_{n-2} = 0$$

Indicial eq: set $n = 0$:

$$a_0 \left[r^2 + 4r + 3\right] = 0 \Rightarrow r^2 + 4r + 3 = 0 \Rightarrow r = -1, -3$$

Case 3: Two distinct roots differing by an integer — this case does not yield two linearly independent solutions. The second solution involves a logarithmic term.

(b) Use $r = -1$ since $-1 > -3$. Recurrence becomes:

$$a_n = -\frac{4a_{n-2}}{(n-1)^2 + 4(n-1) + 3}$$

Start with a_0 free, $a_1 = 0$. Then:

$$\begin{aligned} a_0 &= a_0 \\ a_1 &= a_3 = a_5 = 0 \cdots \\ a_2 &= -\frac{4a_0}{1+4+3} = -\frac{4a_0}{8} = -\frac{a_0}{2} \\ a_4 &= -\frac{4a_2}{9+12+3} = \frac{2a_0}{24} = \frac{a_0}{12} \end{aligned}$$

So first six:

$$a_0, \quad 0, \quad -\frac{a_0}{2}, \quad 0, \quad \frac{a_0}{12}, \quad 0$$

Problem 3:

(a) Boundary conditions that cause energy to dissipate:

Use Dirichlet conditions:

$$u(0,t) = u(L,t) = 0.$$

Heat escapes at the ends, so total energy decreases over time.

(b) To conserve energy:

Use Neumann conditions:

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0.$$

No heat flow across the ends, so energy stays constant.

(c) Given

$$u(x,0) = f(x) = \begin{cases} 1 & 0 < x < L/2, \\ -1 & L/2 < x < L, \end{cases} \quad \text{and} \quad u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$$

Since this is a cosine series, it corresponds to Neumann boundaries. The general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}.$$

Calculate coefficients:

$$A_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{1}{L}\left(\int_0^{L/2} 1\,dx + \int_{L/2}^{L}(-1)dx\right) = 0.$$

For $n \geq 1$:

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx - \int_{L/2}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\right).$$

Evaluate the integrals:

$$\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx = \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right),$$

$$\int_{L/2}^{L}\cos\left(\frac{n\pi x}{L}\right)dx = -\frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

So,

$$A_n = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

This is zero for even n, nonzero for odd n. For odd n, we have

$$A_n = \frac{4}{n\pi}(-1)^{\frac{n-1}{2}}.$$

The final solution is

$$u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi}(-1)^{\frac{n-1}{2}} \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}.$$

Problem 4:

(a) The symmetric Fourier transform is

$$\mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx.$$

Calculate

$$\mathcal{F}[xf(x)](w) = \frac{1}{\sqrt{2\pi}} \int xf(x)e^{-iwx}dx.$$

Differentiating $\mathcal{F}[f]$ w.r.t. w:

$$\frac{d}{dw}\mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int f(x)(-ix)e^{-iwx}dx = -i\mathcal{F}[xf(x)](w).$$

Hence,

$$\mathcal{F}[xf(x)](w) = \frac{1}{-i}\frac{d}{dw}\mathcal{F}[f](w),$$

so $p = i$.

(b) Airy equation:

$$y'' - xy = 0.$$

Taking Fourier transform, denoting $Y(w) = \mathcal{F}[y](w)$:

$$\mathcal{F}[y''] = -w^2 Y, \quad \mathcal{F}[xy] = p\frac{dY}{dw}.$$

The ODE transforms to

$$-w^2 Y + p\frac{dY}{dw} = 0 \to \frac{dY}{dw} = \frac{w^2}{-p}Y.$$

Solve by separation:

$$\frac{dY}{Y} = -\frac{w^2}{p}dw \Longrightarrow Y = C\exp\left(-\frac{w^3}{3p}\right).$$

The inverse Fourier transform is

$$y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Y(w) e^{iwx} dw = \frac{C}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(iwx - \frac{w^3}{3p} \right) dw.$$

Since $y(x)$ is real, we write the integral as the sum of its real and imaginary parts. Using the symmetry properties, the imaginary parts cancel and the integral reduces to a cosine integral over positive w:

$$\mathrm{Ai}(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \cos\left(Aw^3 + Bxw \right) dw,$$

where

$$A = \frac{1}{3|p|} \quad \text{and} \quad B = 1.$$

Because $p = i$ has magnitude 1, we have

$$A = \frac{1}{3}, \quad B = 1.$$