Quantum Physics 1 - Homework 1

Solutions

1. [0.2pt.×10 = 2pt.] Which of the following wave functions below can correspond to a physically realisable state for a particle if $A \in \mathbb{R}_{++}$ is some constant?

In order to have a wave function, which can represent a physical state, the wave function has to be normalisable. Therefore, it should also be square integrable, meaning that $\int |\psi(x)|^2 dx < \infty$. A lack of a continuous derivative can also be accepted as a valid point for c) and f) , as in order to satisfy the S.E. (differential equation) the wavefunction needs to be differentiable. For for i) we note that the limit of $\psi(x)$ as $x \to 0$ does exists (=1).

a. $\psi(x) = A$;

b. $\psi(x) = Ae^{-x}$;

c. $\psi(x) = Ae^{-|x|}$;

d. $\psi(x) = Ae^{-x^2}$;

e. $\psi(x) = A/x$ on $x \in [1, \infty)$;

f. $\psi(x) = A/\sqrt{x}$ on $x \in [1, \infty)$;

g. $\psi(x) = Ae^{-\log(x)^2}$ on $x \in (0, \infty)$;

h. $\psi(x) = A\sin(x)$ on $x \in [-\pi, \pi]$;

i. $\psi(x) = A\sin(x)/x$;

j. $\psi(x) = A[\cos(x) + i\sin(x)]$

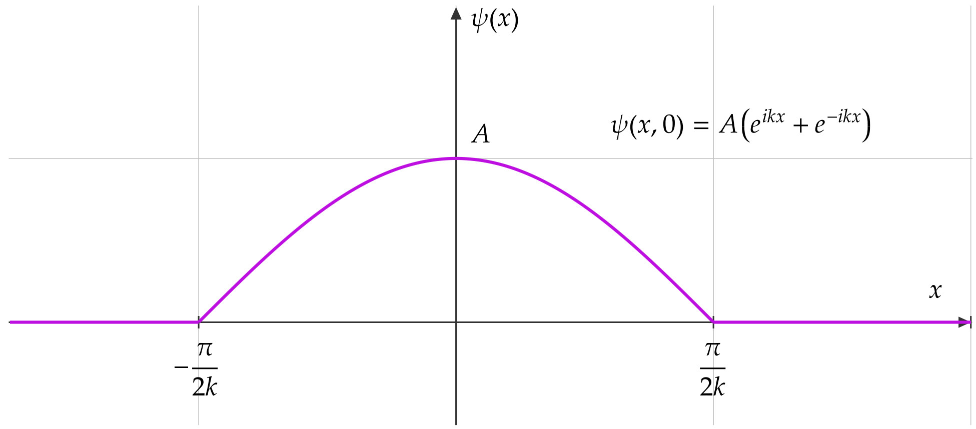
2. [8pt.] Consider a wave function of a particle of mass m at $t = 0$ given by

$$\Psi(x, 0) = \begin{cases} A(e^{ikx} + e^{-ikx}) & \text{if } -\frac{\pi}{2k} \leq x \leq \frac{\pi}{2k} \\ 0 & \text{otherwise} \end{cases}.$$

a. [1pt.] *Find the normalisation constant A and sketch the wave function.*

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 dx \\ &= \int_{-\pi/2k}^{\pi/2k} |2A\cos(kx)|^2 dx = 4|A|^2 \int_{-\pi/2k}^{\pi/2k} \cos^2(kx) dx \\ &= 4|A|^2 \frac{\pi}{2\,k} = |A|^2 \frac{2\pi}{k} \\ |A|^2 &= \frac{k}{2\pi} \to \boxed{A = \sqrt{\frac{k}{2\pi}}} \end{aligned}$$

Note that it should be $2A$ on the vertical axis.



b. [1pt.] *What is the probability that the particle can be found on the interval* $0 \leq x \leq \frac{\pi}{2k}$*?*

$P(0 \leq x \leq \frac{\pi}{2k}) = \frac{1}{2}$, which can be seen directly from the sketch since $x = 0$ is a spatial symmetry axis of the wave function.

c. [2pt.] *Calculate the standard deviation of* x*;* σ_x*.*
$\langle x \rangle = 0$, as the wavefunction is spatially symmetric on the $x = 0$ axis. This can also be calculated using

$$\langle x \rangle = \int \psi^*(x) x \psi(x) dx = 4|A|^2 \int_{-\pi/2k}^{\pi/2k} x \cos^2(kx) dx = 0$$

as this is an odd integrand on integrated over even boundaries

$$\langle x^2 \rangle = \int \psi^*(x) x^2 \psi(x) dx = 4|A|^2 \int_{-\pi/2k}^{\pi/2k} x^2 \cos^2(kx) dx$$

$$= 4|A|^2 \left[\frac{1}{2} \underbrace{\int_{-\pi/2k}^{\pi/2k} x^2 \cos(2kx) dx}_{\text{Integrate by parts x2 or use integral calculator}} + \frac{1}{2} \int_{-\pi/2k}^{\pi/2k} x^2 dx \right] = \frac{\pi^2 - 6}{12k^2}$$

$$\implies \boxed{\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\pi^2 - 6}{12k^2}} = \frac{1}{k}\sqrt{\frac{\pi^2 - 6}{12}} \approx \frac{0.57}{k}}$$

d. [2pt.] *Calculate the standard deviation of momentum* p*;* σ_p*.*

Since we only know how the wavefunction looks like at time $t = 0$ we cannot use $\langle p \rangle = \frac{d\langle x \rangle}{dt}$.

$$\langle p \rangle = -i\hbar \int \psi^*(x) \frac{\partial \psi(x)}{\partial x} dx = -i\hbar 4A^2 k \int_{-\pi/2k}^{\pi/2k} \sin(kx)\cos(kx) dx = -i\hbar 4A^2 \frac{k}{2} \int_{-\pi/2k}^{\pi/2k} \sin(2kx) dx = \boxed{0}.$$

$$\langle p^2 \rangle = -\hbar^2 \int \psi^*(x) \frac{\partial^2 \psi(x)}{\partial x^2} dx = \hbar^2 k^2 \underbrace{4A^2 \int_{-\pi/2k}^{\pi/2k} \cos^2(kx) dx}_{\int |\psi(x)|^2 dx = 1} = \boxed{\hbar^2 k^2}. \implies \boxed{\sigma_p = \hbar k}$$

e. [1pt.] *Does this particle obey Heisenberg's famous position-momentum uncertainty principle?*

From parts c. and d. we find $\sigma_x \sigma_p \approx 0.57\,\hbar > \frac{\hbar}{2}$. Hence, our wave function obeys Heisenberg's uncertainty principle!

f. [1pt.] *Find the expectation value of the kinetic energy of the particle $\langle E \rangle$. How is the spatial spread of the wave function related to $\langle E \rangle$? How can you explain the dependence, which you find?*

$$\langle E \rangle = \left\langle \frac{p^2}{2m} \right\rangle = \frac{\langle p^2 \rangle}{2m} = \boxed{\frac{\hbar^2 k^2}{2m}}.$$

The physical size of the wave function is $d = \pi/k$ and therefore $\langle E \rangle \propto \frac{1}{d^2}$, which leads to the conclusion that our wave function gains energy when it is confined to a smaller physical space. This phenomenon can be linked to the uncertainty principle. A more confined wave function means we have more restricted knowledge about the position of the particle, which the wave function represents (*Note:* $\sigma_x \propto d$). Due to the uncertainty principle, this directly has to result in a broader distribution of possible momenta of the particle, which in turn leads to a higher expectation value for (kinetic) energy.

Grade = 10!