

Quantum Physics 1 - Homework 3

Solutions

Double Delta-Function Potential [10.5pts.]

Grading note: Up to 10.5pts can be obtained in this homework.

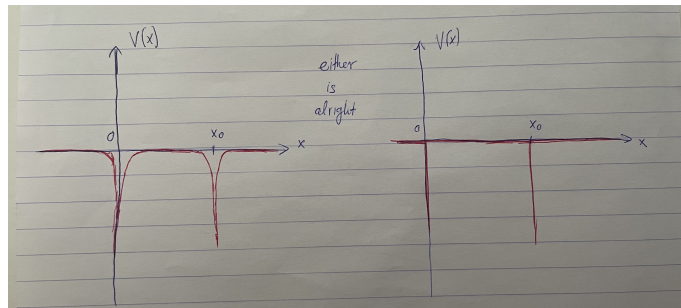
Grade	Points
1	0 - 2.5
4	3 - 5.5
7	6 - 8.5
10	9 - 10.5

We are going to take a look at some of the possible behaviours of a particle of mass m in a one-dimensional double-delta potential given by:

$$V(x) = -\alpha\delta(x) - \alpha\delta(x - x_0), \quad (1)$$

where α and x_0 are positive real numbers.

1. (0.5pts.) Sketch this potential.



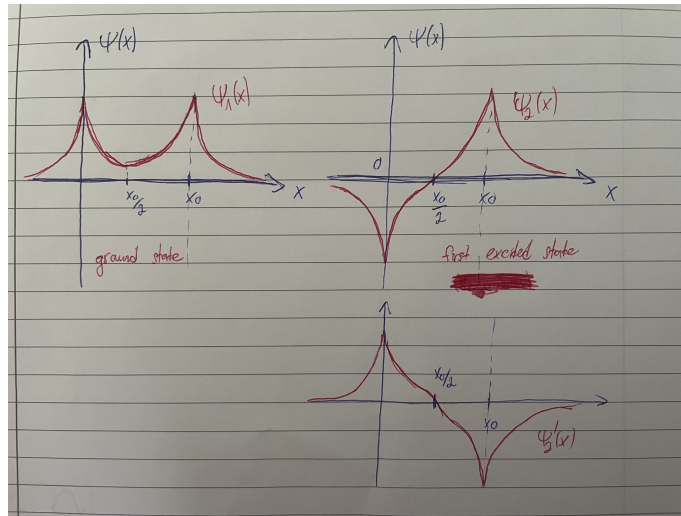
2. (1.5pts.) How many bound states with different energies are permitted in the potential given by equation 1? Sketch all options and label them $\psi_1, \psi_2, \dots$. (You don't have to find the mathematical description of their wavefunctions.)

There can be up to two bound states of different energy - a ground state ψ_1 and an excited state ψ_2 . More in exercise 2.27 in 2nd Ed.

This question was often solved quantitatively instead of qualitatively, causing it to take way longer than expected.

3. (1pt.) In which points or intervals is there a non-zero probability of finding a particle in a bound state? (You can discuss each bound state individually if necessary.) How does this compare to a classical particle?

The ground state has a non-zero probability everywhere (although it's exponentially decreasing), the probability distribution for the first excited state is non-zero everywhere except for $x = x_0/2$. A bound classical particle in such potential would be allowed to exist only in either of the wells (i.e. at the points $x = 0$ or $x = x_0$).



4. (0.5pts.) What range of energies do the scattering states correspond to?
 The scattering states can have any energy larger than 0.

In the next few steps, we will derive the reflection and transmission coefficients for some of the states. Consider scattering states that have a wavenumber (and hence momentum) given by $k = n\pi/x_0$ with n an arbitrary positive integer.

5. (1.5pts.) Write down the most general spatial wavefunction corresponding to an incoming scattering state **from the left** (i.e. from negative infinity). Is this wavefunction normalisable? Do not worry about time-dependence, boundary conditions and calculating normalisation for the moment.

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & \text{if } x < 0 \\ Ce^{ikx} + De^{-ikx}, & \text{if } 0 \leq x \leq x_0 \\ Ee^{ikx}, & \text{if } x > x_0 \end{cases} \quad (2)$$

This wavefunction is not normalisable, just as the free particle stationary states are not.

6. (1.5pts.) What boundary conditions need to be imposed?
- (a) The wave function must be continuous everywhere. Complex exponentials (sines and cosines) are continuous on their domain, so we must now only pay attention to the points where domains of definition connect:
 $x = 0$:

$$A + B = C + D$$

$x = x_0$:

$$Ce^{ikx_0} + De^{-ikx_0} = Ee^{ikx_0}$$

$$Ce^{in\pi} + De^{-in\pi} = Ee^{in\pi}, \text{ using } k = \frac{n\pi}{x_0}$$

$$C + D = E$$

Combined we obtain condition: $A + B = C + D = E$

- (b) The derivative of the wavefunction must be continuous everywhere except for points of infinite potential. In such cases, a condition resulting from a size of a jump in the derivative can be calculated in the following way, starting from the time-independent Schrödinger equation:

$x = 0$:

$$\begin{aligned} & -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x) \\ & -\int_{-\epsilon}^{\epsilon} \frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} dx + \int_{-\epsilon}^{\epsilon} V(x)\psi(x) dx = \int_{-\epsilon}^{\epsilon} E\psi(x) dx \\ & -\frac{\hbar^2}{2m} \int_{-\epsilon}^{\epsilon} \frac{\partial^2 \psi(x)}{\partial x^2} dx - \alpha \int_{-\epsilon}^{\epsilon} \delta(x)\psi(x) dx = \int_{-\epsilon}^{\epsilon} E\psi(x) dx \end{aligned}$$

Now take the limit $\epsilon \rightarrow 0$:

$$\begin{aligned} & -\frac{\hbar^2}{2m} \Delta \frac{\partial \psi(x)}{\partial x} \Big|_{x=0} - \alpha \psi(0) = 0 \\ & \Delta \frac{\partial \psi(x)}{\partial x} \Big|_{x=0} = -\frac{2m\alpha}{\hbar^2} \psi(0) \end{aligned}$$

To find the jump in the derivative, we look at the values of the derivative from left and from right and then take their difference:

$$\begin{aligned} \frac{\partial \psi(x)}{\partial x} \Big|_{x=0^-} &= Aike^{ikx} - Bike^{-ikx} \Big|_{x=0} = ik(A - B) \\ \frac{\partial \psi(x)}{\partial x} \Big|_{x=0^+} &= Cike^{ikx} - Dike^{-ikx} \Big|_{x=0} = ik(C - D) \end{aligned}$$

Plugging the value of ψ at 0 and the difference back into previous equation yields:

$$\begin{aligned} ik(C - D - A + B) &= -\frac{2m\alpha}{\hbar^2} (A + B) \\ C - D - A + B &= -i\frac{2m\alpha}{k\hbar^2} (A + B) \\ C - D &= A \left(1 + i\frac{2m\alpha}{k\hbar^2}\right) - B \left(1 - i\frac{2m\alpha}{k\hbar^2}\right) \end{aligned}$$

Similar procedure can be applied to the point $x = x_0$, this time using the jump in derivatives between $Ce^{ikx} + De^{-ikx}$ and Ee^{ikx} . This obtains the following boundary condition:

$$C - D = E \left(1 - i\frac{2m\alpha}{k\hbar^2}\right)$$

Combining the two boundary conditions from the derivative of ψ yields:

$$A \left(1 + i\frac{2m\alpha}{k\hbar^2}\right) - B \left(1 - i\frac{2m\alpha}{k\hbar^2}\right) = C - D = E \left(1 - i\frac{2m\alpha}{k\hbar^2}\right)$$

7. (2pts.) Now that you have the potential, wavefunction and boundary conditions, find the transmission and reflection coefficients as a function of α .

$$T = \frac{|E|^2}{|A|^2} \quad \text{and} \quad R = \frac{|B|^2}{|A|^2}$$

We use the boundary conditions to find the relation between E and A , and B and A .

$$\begin{aligned} A \left(1 + i \frac{2m\alpha}{k\hbar^2}\right) - B \left(1 - i \frac{2m\alpha}{k\hbar^2}\right) &= E \left(1 - i \frac{2m\alpha}{k\hbar^2}\right) \\ A \left(1 + i \frac{2m\alpha}{k\hbar^2}\right) - B \left(1 - i \frac{2m\alpha}{k\hbar^2}\right) &= (A + B) \left(1 - i \frac{2m\alpha}{k\hbar^2}\right) \\ A \left(1 + i \frac{2m\alpha}{k\hbar^2}\right) - B \left(1 - i \frac{2m\alpha}{k\hbar^2}\right) &= A \left(1 - i \frac{2m\alpha}{k\hbar^2}\right) + B \left(1 - i \frac{2m\alpha}{k\hbar^2}\right) \\ A 2i \frac{2m\alpha}{k\hbar^2} &= B \left(2 - 2i \frac{2m\alpha}{k\hbar^2}\right) \\ A i \frac{2m\alpha}{k\hbar^2} &= B \left(1 - i \frac{2m\alpha}{k\hbar^2}\right) \\ \frac{B}{A} &= \frac{i \frac{2m\alpha}{k\hbar^2}}{1 - i \frac{2m\alpha}{k\hbar^2}} \end{aligned}$$

$$R = \frac{|B|^2}{|A|^2} = \frac{\left|i \frac{2m\alpha}{k\hbar^2}\right|^2}{\left|1 - i \frac{2m\alpha}{k\hbar^2}\right|^2} = \frac{\frac{4m^2\alpha^2}{k^2\hbar^4}}{1 + \frac{4m^2\alpha^2}{k^2\hbar^4}} = \frac{4m^2\alpha^2}{k^2\hbar^4 + 4m^2\alpha^2}$$

We can substitute for k here and continue as below but it is not necessary.

$$R = \frac{4m^2\alpha^2}{\frac{n^2\pi^2}{x_0^2}\hbar^4 + 4m^2\alpha^2} = \frac{4m^2\alpha^2 x_0^2}{n^2\pi^2\hbar^4 + 4m^2\alpha^2 x_0^2}$$

Same method can be used to obtain the transmission coefficient, at the beginning we substitute $B = E - A$:

$$A \left(1 + i \frac{2m\alpha}{k\hbar^2}\right) - (E - A) \left(1 - i \frac{2m\alpha}{k\hbar^2}\right) = E \left(1 - i \frac{2m\alpha}{k\hbar^2}\right).$$

After a bit of rearranging we can find the transmission coefficient to be

$$T = \frac{|E|^2}{|A|^2} = \frac{1}{1 + \frac{4m^2\alpha^2}{k^2\hbar^4}} = \frac{k^2\hbar^4}{k^2\hbar^4 + 4m^2\alpha^2} \left(= \frac{\frac{n^2\pi^2}{x_0^2}\hbar^4}{\frac{n^2\pi^2}{x_0^2}\hbar^4 + 4m^2\alpha^2} = \frac{n^2\pi^2\hbar^4}{n^2\pi^2\hbar^4 + 4m^2\alpha^2 x_0^2} \right).$$

Many students did not notice the value given for k and solved the problem for a general case, which requires significantly more effort and time.

8. (1pt.) In general, do the scattering states of this potential have discrete energy levels? In the questions above we set $k = n\pi/x_0$ with $n \in \mathbb{N}$. Is this the correct/only possible quantisation of the energies of the scattering states?

Scattering states of this system do not have discrete energy levels but instead make up a continuous spectrum. (In the calculation of the transmission and reflection coefficients, we merely chose a subset of the stationary states which have wavenumber k easy to work with.)

An arbitrary wavefunction can be written as a linear combination of functions from a complete set of eigenstates.

9. (1pt.) Write down the general formula for an arbitrary wavefunction $\Psi(x, t)$ using the stationary states of the potential given by equation 1. Don't forget to include time dependence. You don't have to find the exact energies and you can assume that the stationary states of this potential form a complete set. *Hint: Check out equation 2.100 (2nd Ed) or 2.101 (3rd Ed)*

$$\Psi(x, t) = c_1 \psi_1(x) e^{-iE_1 t/\hbar} + c_2 \psi_2(x) e^{-iE_2 t/\hbar} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-iE(k)t/\hbar} \phi(k) dk$$

Only very few students included the bound states.

Grade = 10 × percentage of your points rounded to the nearest integer in {1, 4, 7, 10}.