Quantum Physics 1 - Homework 5

Due on Thursday (Sept 9) at 23:59

Grade	Points
1	0 - 1.5
4	2 - 3.5
7	4 - 7
10	7.5 - 9

1. [2pt.] Are the following statements true or false?

(a) A state cannot be a simultaneous eigenstate of two compatible observables. False

(b) Hermitian operators have real eigenvalues in finite-dimensional vector spaces. True

(c) All wavefunctions are linear combinations of eigenfunctions of a Hermitian operator. True

(d) The generalized uncertainty principle is one of the central assumptions of quantum mechanics. False

(e) If a matrix is equal to the complex conjugate of its transpose, then the matrix is Hermitian. True

(f) All observables are Hermitian. True

2. [3pt] Recall that the groundstate wavefunction for the harmonic oscillator is given by:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \tag{1}$$

Calculate the expectation value of the product of the position and momentum operator, $\langle xp \rangle$, in the ground state of the harmonic oscillator. What does the answer tell you about the hermiticity of the $\hat{x}\hat{p}$ operator?

The expectation value can be calculated as follows:

$$\begin{aligned}
\langle xp \rangle &= \int_{-\infty}^{\infty} \psi_0(x) \left(-i\hbar x \frac{d}{dx}\right) \psi_0(x)\, dx && (2)\\
&= -i\hbar \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \frac{d}{dx} e^{-\frac{m\omega}{2\hbar}x^2}\, dx && (3)\\
&= im\omega \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}\, dx && (4)\\
&= i\hbar/2 && (5)
\end{aligned}$$

Because the expectation value is imaginary, we conclude that xp cannot be a hermitian operator.

2. [2+1+1= 4pt.] The Hamiltonian for a certain three-level system is

$$\hat{H} = \epsilon \left(|\alpha\rangle \langle\alpha| + |\alpha\rangle \langle\gamma| + |\beta\rangle \langle\beta| + |\gamma\rangle \langle\alpha| + |\gamma\rangle \langle\gamma| \right), \tag{6}$$

where $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ is an orthonormal basis and ϵ is a nonzero number with the dimensions of energy.

(a) Taking $\hat{H}|\alpha\rangle = \epsilon(|\alpha\rangle + |\beta\rangle)$ tells us that $|\alpha\rangle$ is **not** an eigenstate of the Hamiltonian. Find the normalised eigenstates and corresponding eigenvalues of the Hamiltonian. *(Hint: Write the general eigenvector as $|\psi\rangle = c_\alpha |\alpha\rangle + c_\beta |\beta\rangle + c_\gamma |\gamma\rangle$)*

Write the eigenvector as $|\psi\rangle = c_\alpha |\alpha\rangle + c_\beta |\beta\rangle + c_\gamma |\gamma\rangle$, and call the eigenvalue E. The eigenvalue equation is

$$\hat{H}|\psi\rangle = \epsilon \left(|\alpha\rangle \langle\alpha| + |\alpha\rangle \langle\gamma| + |\beta\rangle \langle\beta| + |\gamma\rangle \langle\alpha| + |\gamma\rangle \langle\gamma| \right) \left(c_\alpha |\alpha\rangle + c_\beta |\beta\rangle + c_\gamma |\gamma\rangle \right)$$
$$\hat{H}|\psi\rangle = \epsilon \left((c_\alpha + c_\gamma) |\alpha\rangle + c_\beta |\beta\rangle + (c_\alpha + c_\gamma) |\gamma\rangle \right)$$

Where we have used the orthonormality of the basisvectors $\langle f_m | f_n \rangle = \delta_{mn}$. We find 3 sets of equations:

$$\epsilon(c_\alpha + c_\gamma)|\alpha\rangle = E c_\alpha |\alpha\rangle \tag{7}$$
$$\epsilon \left(c_\beta |\beta\rangle \right) = E c_\beta |\beta\rangle \tag{8}$$
$$\epsilon(c_\alpha + c_\gamma)|\beta\rangle = E c_\gamma |\gamma\rangle \tag{9}$$

From 7 and 9, we see that either $c_\alpha = c_\gamma$ or $E = 0$ in which case $c_\alpha = -c_\gamma$. The latter can be normalized to give the first normalized eigenstate

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \left[|\alpha\rangle - |\gamma\rangle \right].$$

Meanwhile, when $c_\alpha = c_\gamma$, we can let $c_\beta = 0$ to find another eigenvector, this time with eigenvalue 2ϵ. Normalizing this eigenstate gives

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \left[|\alpha\rangle + |\gamma\rangle \right].$$

Finally, by setting $c_\alpha = c_\gamma = 0$, we can let $c_\beta = 1$ to obtain a third eigenstate with eigenvalue ϵ

$$|\Psi_3\rangle = |\beta\rangle.$$

Note that $|\psi_2\rangle$ is not an excited state of $|\psi_1\rangle$, it is just the notation used to number the eigenstates.

(b) What is the matrix **H** representing $\hat{H}$ with respect to the $\{|\alpha\rangle, |\beta\rangle, |\gamma\rangle\}$ basis?

$$\boxed{\mathbf{H} = \epsilon \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$$

(c) Denote the time evolution of the eigenstates.

$$\boxed{\begin{gathered} \big|\Psi_1(t)\big\rangle = \tfrac{1}{\sqrt{2}}\left[|\alpha\rangle - |\gamma\rangle\right]. \\ \big|\Psi_2(t)\big\rangle = \tfrac{1}{\sqrt{2}}\left[|\alpha\rangle + |\gamma\rangle\right] e^{+i2\epsilon t/\hbar} \\ \big|\Psi_3(t)\big\rangle = |\beta\rangle\, e^{+i\epsilon t/\hbar} \end{gathered}}$$

also accepted

$$\big|\Psi(t)\big\rangle = c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle\, e^{+i2\epsilon t/\hbar} + c_3\,|\psi_3\rangle\, e^{+i\epsilon t/\hbar}.$$

$Grade = 10.$