Quantum Physics 1 - Homework 7

Due on Monday (Oct 24th) at 11:59 AM

Grade	Points
1	0 - 2.5
4	3 - 5
7	5.5 - 7.5
10	8 - 9.5

1a) [1pt.] Show that the operators $\hat{A}$ and $\hat{B}$ commute if the matrices A and B representing them have the same eigenvectors, and these eigenvectors span the vector space.

Let's call the eigenvalues of our matrices a and b, such that

$$A\mathbf{v} = a\mathbf{v}, \tag{1}$$

and

$$B\mathbf{v} = b\mathbf{v}, \tag{2}$$

where $\mathbf{v}$ is an eigenvector of the two matrices. If we apply the matrices successively we get

$$BA\mathbf{v} = ba\mathbf{v}. \tag{3}$$

Since $ba = ab$ we can rewrite this to

$$BA\mathbf{v} = ab\mathbf{v}. \tag{4}$$

This is only true when $BA = AB$, if this holds for every eigenvector this gives us that (hence the question states that all eigenvectors are the same, not just one)

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0. \tag{5}$$

b) [0.5pt.] What does your result in a) tell you about the value of the commutator $\left[\hat{S}^2, \hat{S}_z\right]$?

eq 4.142 and 4.144 tell us that the matrices representing the operators have the same eigenspinors, so they commute,

$$\left[\hat{S}^2, \hat{S}_z\right] = 0. \tag{6}$$

c) [1pt.] What is the trace of the Pauli matrices? And how does it relate to their eigenvalues?

$$tr(\sigma_x) = tr(\sigma_y) = tr(\sigma_z) = 0 = \sum_i \lambda_i \tag{7}$$

The trace of the matrices and sum of the eigenvalues is zero, this also holds for a spin 3/2 particle as you can see in q2bc.

2a) [0.5pt.] What are the possible states $|sm\rangle$ for a particle with s = 3/2?

$$|sm\rangle = \left|\frac{3}{2} - \frac{3}{2}\right\rangle, \left|\frac{3}{2} - \frac{1}{2}\right\rangle, \left|\frac{3}{2}\frac{1}{2}\right\rangle, \left|\frac{3}{2}\frac{3}{2}\right\rangle \tag{8}$$

b) [1pt.] Use the results of exercise 4.53 (2nd edition) or 4.62 (3rd edition) to construct the matrices representing $\hat{S}_x$ and $\hat{S}_z$ for a particle with $s = 3/2$.

$$S_z = \hbar \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix} \tag{9}$$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{10}$$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & b_s & 0 & 0 \\ b_s & 0 & b_{s-1} & 0 \\ 0 & b_{s-1} & 0 & b_{s-2} \\ 0 & 0 & b_{s-2} & 0 \end{pmatrix} \tag{11}$$

$s = 3/2$, so $b_j = \sqrt{(3/2 + j)(5/2 - j)}$.

$$b_s = \sqrt{3}$$
$$b_{s-1} = 2$$
$$b_{s-2} = \sqrt{3}$$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \tag{12}$$

c) [1.5pt.] What are the eigenvalues of the matrix S_x you constructed in the previous question? If you didn't get an answer in b) you can use S_y instead:

$$S_y = \frac{i\hbar}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \tag{13}$$

To find the eigenvalues λ we will use the characteristic equation:

$$det\,(S_x - \lambda I) = 0 \tag{14}$$

$$\begin{vmatrix} -\lambda & \frac{\sqrt{3}\hbar}{2} & 0 & 0 \\ \frac{\sqrt{3}\hbar}{2} & -\lambda & \hbar & 0 \\ 0 & \hbar & -\lambda & \frac{\sqrt{3}\hbar}{2} \\ 0 & 0 & \frac{\sqrt{3}\hbar}{2} & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & \hbar & 0 \\ \hbar & -\lambda & \frac{\sqrt{3}\hbar}{2} \\ 0 & \frac{\sqrt{3}\hbar}{2} & -\lambda \end{vmatrix} - \frac{\sqrt{3}\hbar}{2} \begin{vmatrix} \frac{\sqrt{3}\hbar}{2} & \hbar & 0 \\ 0 & -\lambda & \frac{\sqrt{3}\hbar}{2} \\ 0 & \frac{\sqrt{3}\hbar}{2} & -\lambda \end{vmatrix} \tag{15}$$

$$= \lambda^2 \begin{vmatrix} -\lambda & \frac{\sqrt{3}\hbar}{2} \\ \frac{\sqrt{3}\hbar}{2} & -\lambda \end{vmatrix} + \lambda\hbar \begin{vmatrix} \hbar & \frac{\sqrt{3}\hbar}{2} \\ 0 & -\lambda \end{vmatrix} - \frac{3\hbar^2}{4} \begin{vmatrix} -\lambda & \frac{\sqrt{3}\hbar}{2} \\ \frac{\sqrt{3}\hbar}{2} & -\lambda \end{vmatrix} + \frac{\sqrt{3}\hbar^2}{2} \begin{vmatrix} 0 & \frac{\sqrt{3}\hbar}{2} \\ 0 & -\lambda \end{vmatrix} \tag{16}$$

$$= \lambda^4 - \frac{3\hbar^2\lambda^2}{4} - \hbar^2\lambda^2 - \frac{3\hbar^2\lambda^2}{4} + \frac{9\hbar^4}{16} = \lambda^4 - \frac{5\hbar^2\lambda^2}{2} + \frac{9\hbar^4}{16} = 0 \tag{17}$$

By substituting $x = \lambda^2$ we can use the quadratic equation to find: $\lambda_1 = \frac{3\hbar}{2}$, $\lambda_2 = \frac{\hbar}{2}$, $\lambda_3 = -\frac{\hbar}{2}$ and $\lambda_4 = -\frac{3\hbar}{2}$.

For S_y the same process can be used to obtain

$$\begin{vmatrix} -\lambda & -\frac{\sqrt{3}\hbar}{2} & 0 & 0 \\ \frac{\sqrt{3}\hbar}{2} & -\lambda & -\hbar & 0 \\ 0 & \hbar & -\lambda & -\frac{\sqrt{3}\hbar}{2} \\ 0 & 0 & \frac{\sqrt{3}\hbar}{2} & -\lambda \end{vmatrix} = \lambda^4 - \frac{5\hbar^2\lambda^2}{2} + \frac{9\hbar^4}{16}. \tag{18}$$

By again substituting $x = \lambda^2$ we find that $\lambda_1 = \frac{3\hbar}{2}$, $\lambda_2 = \frac{\hbar}{2}$, $\lambda_3 = -\frac{\hbar}{2}$ and $\lambda_4 = -\frac{3\hbar}{2}$.

d) [0.5pt.] Explain qualitatively why the eigenvalues of S_x, S_y and S_z have to be the same.

The matrices S_x, S_y and S_z all describe a spin 3/2 particle and the coordinate axes are chosen arbitrarily, meaning there is no preferred direction. So it wouldn't make any sense for matrices to have different eigenvalues.

e) [2pt.] Find the normalised eigenspinors of S_x, you can use S_y if you didn't find an answer in b).

$$\begin{pmatrix} 0 & \frac{\sqrt{3}\hbar}{2} & 0 & 0 \\ \frac{\sqrt{3}\hbar}{2} & 0 & \hbar & 0 \\ 0 & \hbar & 0 & \frac{\sqrt{3}\hbar}{2} \\ 0 & 0 & \frac{\sqrt{3}\hbar}{2} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}\hbar}{2}b \\ \frac{\sqrt{3}\hbar}{2}a + \hbar c \\ \hbar b + \frac{\sqrt{3}\hbar}{2}d \\ \frac{\sqrt{3}\hbar}{2}c \end{pmatrix} \tag{19}$$

$$\lambda_n \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}\hbar}{2}b \\ \frac{\sqrt{3}\hbar}{2}a + \hbar c \\ \hbar b + \frac{\sqrt{3}\hbar}{2}d \\ \frac{\sqrt{3}\hbar}{2}c \end{pmatrix} \tag{20}$$

We will normalise the eigenspinor later anyway, so let's pick $a = 1$ and solve the equation for the eigenvalues we found in c), starting with $\lambda_1 = \frac{3\hbar}{2}$:

$$\frac{3\hbar}{2} = \frac{\sqrt{3}\hbar}{2}b$$

$$b = \sqrt{3}$$

$$\frac{3\sqrt{3}\hbar}{2} = \frac{\sqrt{3}\hbar}{2} + \hbar c$$

$$c = \sqrt{3}$$

$$\frac{3\hbar}{2}d = \frac{\sqrt{3}\hbar}{2}\sqrt{3}$$

$$d = 1$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{pmatrix} \tag{21}$$

$$\hat{v}_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{pmatrix} \tag{22}$$

Now for $\lambda_2 = \frac{\hbar}{2}$

$$\frac{\hbar}{2} = \frac{\sqrt{3}\hbar}{2}b$$

$$b = \frac{1}{\sqrt{3}}$$

$$\frac{\hbar}{2\sqrt{3}} = \frac{\sqrt{3}\hbar}{2} + \hbar c$$

$$c = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\frac{\hbar d}{2} = -\frac{\sqrt{3}\hbar}{2}\frac{1}{\sqrt{3}}$$

$$d = -1$$

$$\hat{v}_2 = \frac{\sqrt{6}}{4} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -1 \end{pmatrix} \tag{23}$$

For $\lambda_3 = -\frac{\hbar}{2}$

$$-\frac{\hbar}{2} = \frac{\sqrt{3}\hbar}{2}b$$

$$b = -\frac{1}{\sqrt{3}}$$

$$\frac{\hbar}{2\sqrt{3}} = \frac{\sqrt{3}\hbar}{2} + \hbar c$$

$$c = \frac{1}{2\sqrt{3}} - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}$$

$$-\frac{\hbar d}{2} = -\frac{\sqrt{3}\hbar}{2}\frac{1}{\sqrt{3}}$$

$$d = 1$$

$$\hat{v}_3 = \frac{\sqrt{6}}{4}\begin{pmatrix} 1 \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} \tag{24}$$

For $\lambda_4 = -\frac{3\hbar}{2}$

$$b = -\sqrt{3}$$

$$\frac{3\sqrt{3}\hbar}{2} = \frac{\sqrt{3}\hbar}{2} + \hbar c$$

$$c = \sqrt{3}$$

$$-\frac{3\hbar}{2}d = \frac{\sqrt{3}\hbar}{2}\sqrt{3}$$

$$d = -1$$

$$\hat{v}_4 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{pmatrix} \tag{25}$$

For S_y the process is very similar, you should get

$$\begin{pmatrix} 0 & -i\frac{\sqrt{3}\hbar}{2} & 0 & 0 \\ i\frac{\sqrt{3}\hbar}{2} & 0 & -i\hbar & 0 \\ 0 & i\hbar & 0 & -i\frac{\sqrt{3}\hbar}{2} \\ 0 & 0 & i\frac{\sqrt{3}\hbar}{2} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -i\frac{\sqrt{3}\hbar}{2}b \\ i\frac{\sqrt{3}\hbar}{2}a - i\hbar c \\ i\hbar b - i\frac{\sqrt{3}\hbar}{2}d \\ i\frac{\sqrt{3}\hbar}{2}c \end{pmatrix} \tag{26}$$

$$\lambda_n\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -i\frac{\sqrt{3}\hbar}{2}b \\ i\frac{\sqrt{3}\hbar}{2}a - i\hbar c \\ i\hbar b - i\frac{\sqrt{3}\hbar}{2}d \\ i\frac{\sqrt{3}\hbar}{2}c \end{pmatrix} \tag{27}$$

Picking $a = 1$ again, starting with λ_1 (I have left $\hbar$ out, since it's a constant factor that everything is multiplied by):

$$\frac{3}{2} = -i\frac{\sqrt{3}}{2}b$$

$$b = i\sqrt{3}$$

$$\frac{i3\sqrt{3}}{2} = i\frac{\sqrt{3}}{2} - ic$$

$$c = \sqrt{3}$$
$$\frac{3}{2}d = i\frac{\sqrt{3}}{2}\sqrt{3}$$
$$d = i$$

$$\hat{v}_1 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ i\sqrt{3} \\ \sqrt{3} \\ i \end{pmatrix} \tag{28}$$

Remember to take the complex conjugate when calculating the length of the vector in order to normalise it. For $\lambda_2 = \frac{\hbar}{2}$:

$$\frac{1}{2} = -i\frac{\sqrt{3}}{2}b$$
$$b = \frac{i}{\sqrt{3}}$$
$$\frac{i}{2\sqrt{3}} = i\frac{\sqrt{3}}{2} - ic$$
$$c = \frac{1}{\sqrt{3}}$$
$$\frac{1}{2}d = i\frac{\sqrt{3}}{2\sqrt{3}}$$
$$d = i$$

$$\hat{v}_2 = \frac{\sqrt{6}}{4}\begin{pmatrix} 1 \\ \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ i \end{pmatrix} \tag{29}$$

For $\lambda_3 = -\frac{\hbar}{2}$

$$-\frac{1}{2} = -i\frac{\sqrt{3}}{2}b$$
$$b = -\frac{i}{\sqrt{3}}$$
$$\frac{i}{2\sqrt{3}} = \frac{i\sqrt{3}}{2} - ic$$
$$c = \frac{1}{\sqrt{3}}$$
$$-\frac{d}{2} = i\frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}$$
$$d = -i$$

$$\hat{v}_3 = \frac{\sqrt{6}}{4}\begin{pmatrix} 1 \\ -\frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -i \end{pmatrix} \tag{30}$$

For $\lambda_4 = -\frac{3\hbar}{2}$:

$$-\frac{3}{2} = -i\frac{\sqrt{3}}{2}b$$
$$b = -i\sqrt{3}$$
$$i\frac{3\sqrt{3}}{2} = i\frac{\sqrt{3}}{2} - ic$$
$$c = -\sqrt{3}$$
$$-\frac{3d}{2} = -i\frac{\sqrt{3}}{2}\sqrt{3}$$
$$d = i$$

$$\hat{v}_4 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ -i\sqrt{3} \\ -\sqrt{3} \\ i \end{pmatrix} \tag{31}$$

f) [1.5pt.] The $\left|\frac{3}{2}\frac{3}{2}\right\rangle$ state can be decomposed into the eigenspinors of S_x (or S_y) using

$$\left|\frac{3}{2}\frac{3}{2}\right\rangle = c_1\chi_1^{(x)} + c_2\chi_2^{(x)} + c_3\chi_3^{(x)} + c_4\chi_4^{(x)}, \tag{32}$$

where $\chi_n^{(x)}$ are the eigenspinors of S_x for the eigenvalues λ_n. What are the constants c_n? Check if $\sum |c_n|^2 = 1$. If you didn't get an answer in e) you can use:

$$\chi_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad \chi_2 = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -1 \end{pmatrix} \qquad \chi_3 = \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \qquad \chi_4 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ \sqrt{2} \\ -1 \end{pmatrix}, \tag{33}$$

note that these spinors are not normalised.

$$\left|\frac{3}{2}\frac{3}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{34}$$

Note that these are not technically the same thing, the vector is just a representation of the spin state.
The $\left|\frac{3}{2}\frac{3}{2}\right\rangle$ state is a linear combination of the eigenspinors of S_x (or S_y), so

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = c_1\chi_1 + c_2\chi_2 + c_3\chi_3 + c_4\chi_4. \tag{35}$$

We can multiply by the vector representation of $\left\langle\frac{3}{2}\frac{3}{2}\right|$ to find the coefficients:

$$\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \sum_i^n c_n \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\chi_n = 1. \tag{36}$$

So

$$c_n = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\chi_n, \tag{37}$$

$$c_1 = \frac{1}{2\sqrt{2}}$$

$$c_2 = \frac{\sqrt{6}}{4}$$

$$c_3 = \frac{\sqrt{6}}{4}$$

$$c_4 = \frac{1}{2\sqrt{2}}$$

The solution for the eigenspinors of S_y is the same, since the first component of the eigenspinors is the same as those of S_x. For the eigenspinors given in the question the normalisation constants are

$$\frac{1}{\sqrt{6}}$$

for the 1st and 4th eigenspinors, and

$$\frac{1}{\sqrt{3}}$$

for the 2nd and 3rd eigenspinors.
Now we need to check that the sum of $|c_n|^2$ is 1. For S_x and S_y:

$$\sum_i^n |c_n|^2 = 2\left(\frac{1}{2\sqrt{2}}\right)^2 + 2\left(\frac{\sqrt{6}}{4}\right)^2 = \frac{2}{8} + \frac{2\cdot 6}{16} = 1 \tag{38}$$

For the eigenspinors given in the question:

$$\sum_i^n |c_n|^2 = 2\left(\frac{1}{\sqrt{6}}\right)^2 + 2\left(\frac{1}{\sqrt{3}}\right)^2 = \frac{2}{6} + \frac{2}{3} = 1 \tag{39}$$

$Grade \in \{1, 4, 7, 10\}$.