Quantum Physics 1 - Homework 8
Due on Mon Oct 31, 11:59AM

1. Energy Spectrum of the Hellmann Potential [4×1pt = 4pt.]

In the lecture and the book, the energy spectrum of the Hydrogen atom is derived analytically. Although the Hydrogen atom may be considered as one of the simplest physical systems, it is suprising that the energy spectrum could be found analytically. Finding the energy spectrum for a generic (spherically symmetric) potential is *not* a trivial task at all, and almost always requires complex numerical techniques.

In the literature,[1] a prototypical example of a potential for which the energy spectrum cannot be found analytically is the *Hellmann* potential. This is a combination of the Coulomb potential and the Yukawa potential:

$$V(r) = -\frac{\alpha}{r} + \frac{\beta}{2}\frac{\alpha}{r}\, e^{-\gamma r/a_0}, \tag{1}$$

where the first and second term correspond to the Coulomb and Yukawa potentials.[2] The parameter β (which can be positive or negative) sets the relative importance of the Yukawa term relative to the Coulomb term. The parameter γ, which we take to be positive always, sets the amount of 'screening' of the Yukawa contribution. The generalized Bohr radius is defined as $a_0 \equiv \hbar^2/\mu\alpha$ and energies will be measured in terms of $\epsilon \equiv \mu\alpha^2/2\hbar^2$, where μ is the reduced mass of the system.[3] In these units, the energy spectrum of Hydrogen would be:

$$E_n^{(\mathrm{H})} = -\frac{\epsilon}{n^2}. \tag{2}$$

Instruction for the Animation

Use this link to the animation in Google Colab.[4] The animation shows you the energy spectrum of the Hellmann potential for specific values of β and γ.

- In the first dropdown, you have to choose between a Coulomb (i.e. Hydrogen) and Hellmann (i.e. Coulomb + Yukawa) potential. When you select the former, β is set to zero and changing the values for β and γ will not change the result. You will get the energy spectrum of hydrogen as output.

- If you select the latter, you can choose values for β and γ, and the spectrum will be shown as output. For comparison to the Hydrogen spectrum, use the dotted lines in the left panel. The right panel provides a 'zoom-in', so that the relative positions of the energy levels can be distinguished easily.

Questions

[1] J. Adamowski, *Bound eigenstates of the superposition of the Coulomb and the Yukawa potentials*, APS, 1985.

[2] For the Hydrogen atom, we would have $\alpha = e^2/4\pi\epsilon_0$.

[3] In the case of Hydrogen, this would be the reduced mass of the electron-proton system.

[4] If the resulting web page shows the raw notebook code, click the *Open with Google Colaboratory* button to get the working version in Colab.

a. Take the Coulomb (i.e. Hydrogen-like) potential and consider the spectrum . Argue that the degeneracy of the n-th energy level is n^2. *Hint:* for each value of ℓ, there are $2\ell + 1$ values of m.

   In the hydrogen spectrum, there is only one energy level for $n = 1, \ell = 0$, then there are two energy levels in the spectrum for $n = 2$, namely $\ell = 0$ and $\ell = 1$. These two have degeneracy 1 and 3 respectively. Making the total degeneracy 4 for $n = 2$. For $n = 3$, we have $\ell = 0, 1, 2$, yielding a total degeneracy of 9. Following the pattern we conclude that the degeneracy of the n-th energy level is n^2.

b. Consider the Hellmann potential (Eq. 1). Apart from the obvious limit $\beta \to 0$, find a second limit for which the Hellmann potential reduces to the Coulomb potential. By taking appropriate values for β and γ in the simulation, check that the energy spectrum indeed approaches the Coulomb/Hydrogen-like spectrum.

   In the limit where $\gamma \to \infty$, the Yukawa term becomes negligible and we get back to the Coulomb/Hydrogen-like spectrum. By taking γ large, i.e. 10, we indeed find that the spectrum is almost equivalent to that of Hydrogen.

c. Take $\beta < 0$ and examine what happens with the energy spectrum relative to the Hydrogen-like spectrum. Does $\beta < 0$ correspond to a repulsive or attractive force resulting from the Yukawa contribution to the potential? By reasoning, find out what happens to the expectation value $\langle r \rangle$ of the electron for $\beta < 0$.

   Taking $\beta < 0$ results in the levels shifting down with respect to the hydrogen case. Taking $\beta < 0$ corresponds to an additional attractive force, which classically attracts the electron towards the nucleus. Quantum mechanically, this implies that the expectation value of the electron $\langle r \rangle$ becomes smaller as β becomes smaller (i.e. more negative).

d. Take a generic configuration of the Hellmann potential (e.g. $\beta = 2$, $\gamma = 0.1$). Describe how the spectrum has changed compared the Hydrogen-like spectrum. What is the degeneracy of each energy level in this case? By comparing to (a), in what way is the Hydrogen-like spectrum special?

   For a generic configuration, the degeneracy between different ℓ states (same n) is lost, and the degeneracy of the distinct energy levels is simply $2\ell + 1$. Hydrogen is special, in the sense the energy levels with same n but different ℓ 'line up', yielding a degeneracy of n^2.

2. The EPR Paradox and Bell's Theorem [5 ×1pt = 5 pts.]

In this question, you will examine the EPR-setup. Consider the decay of neutral pion π^0 at the source S into an electron-positron pair. The π^0 has spin zero, requiring the electron-positron pair to be in the singlet configuration:

$$\Psi = \frac{1}{\sqrt{2}}(\uparrow_- \downarrow_+ - \downarrow_- \uparrow_+), \tag{3}$$

where the $\pm$ subscript refers to $e^{\pm}$. We choose our coordinate system in such a way that the electron and positron have their spin aligned along the z-axis. The pion decays at the source S, the electron travels to the left, the positron to the right. The spins of the electron and positron are measured by Alice and Bob, respectively, using spin-detectors independently oriented along unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. These unit vectors make angles θ_a and θ_b with the z-axis as indicated in the schematic below (Fig. 1).

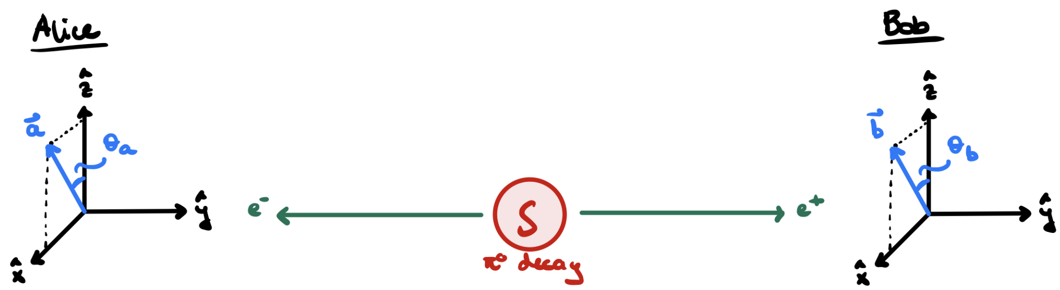

Figure 1: The EPR setup.

The outcomes of the spin-measurements by detectors a and b are denoted as $s_a = \pm 1$ and $s_b = \pm 1$. (We omit the factor of $\hbar/2$ for simplicity.) The product of the spins is denoted as $s_{ab} = s_a \times s_b$. Quantum mechanics predicts the expectation value $\langle s_{ab} \rangle$ to depend solely on the relative orientations of the spin detectors:

$$P(\boldsymbol{a}, \boldsymbol{b}) \equiv \langle s_{ab} \rangle = -\boldsymbol{a} \cdot \boldsymbol{b}. \tag{4}$$

Instruction for the Animation

Use this link to the animation in Google Colab. The simulation allows you to choose the orientation of the two spin-detectors via θ_a and θ_b. In the simulated experiment, we examine N pion-decays and measure the spins of the electron and positron in each case: you can set the value of N. The output of the simulation is a list of N rows and 3 columns (s_a, s_b and s_{ab}): this is option can be toggled on and off. Based on this list, an estimate of the expectation value $\langle s_{ab} \rangle$ is calculated and given as well. Lastly, a bar-chart shows the relative occurrence of the four possible measurement outcomes ($\downarrow\downarrow$, $\downarrow\uparrow$, $\uparrow\downarrow$ and $\uparrow\uparrow$). You can double-click the gray button to redo the simulated experiment.

Questions

a. For what choice(s) of θ_a and θ_b will the product s_{ab} always give the same value? What is this value? Explain briefly.

   For the choice $\theta_a = \theta_b$ the product will always be $s_{ab} = -1$. The reason is that if Alice measures -1, then Bob must measure $+1$ and vice versa.

b. Einstein, Podolsky and Rosen (EPR) considered the setup in (a) to be based on (unsatisfactory) *spooky action on a distance*. Explain why, and make sure to use the term *locality* in your answer.

   The reason is that if Alice measures first (e.g. spin-up), than she knows immediately that Bob must have spin-down (the wavefunction collapsed). In other words, Alice' measurement instantaneously "produced" a spin-down for Bob, no matter how far Alice and Bob are apart: this is known as the spooky-action-on-a-distance.

Suppose that Alice measures first and then Bob, the orientations of the two detectors are arbitrary.

c. What is the probability that Alice measures spin up or spin down as measured along unit vector $\boldsymbol{a}$?

   Alice measures spin-up or down with 50-50 probability, independent of the orientation of $\boldsymbol{a}$.

d. Given are the following two choices for probabilities:

$$P_1 = \sin^2\left(\frac{\theta_a - \theta_b}{2}\right), \qquad P_2 = \cos^2\left(\frac{\theta_a - \theta_b}{2}\right) \tag{5}$$

Given that Alice measures spin up, which of the above two expressions $P_{1,2}$ is the probability that Bob measures spin up as well? Briefly explain why.

We can find out the correct probability by taking limits. In the limit where $\theta_a = \theta_b$, the probability that Bob measures up if Alice did should be zero. This limit is satisfied by P_1. The probability that Bob measures down is given by P_2, which becomes unity as $\theta_a = \theta_b$.

EPR argued that quantum-mechanics could not be the whole story, and that a local hidden variable theory was the correct description instead. However, Bell showed that for *any* local hidden variable theory the *Bell inequality* must hold:

$$|P(\boldsymbol{a}, \boldsymbol{b}) - P(\boldsymbol{a}, \boldsymbol{c})| \leq 1 + P(\boldsymbol{b}, \boldsymbol{c}). \tag{6}$$

e. Find and sketch an orientation of the unit vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ that violates Bell's inequality and therefore rules out local hidden variable theories. Check with the simulation!

A prototype configuration would be that $\boldsymbol{a}$ and $\boldsymbol{b}$ to be orthogonal, and $\boldsymbol{c}$ halfway (at 45 degrees) between them.

Grade = your points+1.