Final Exam Quantum Physics 1 – 2024/2025

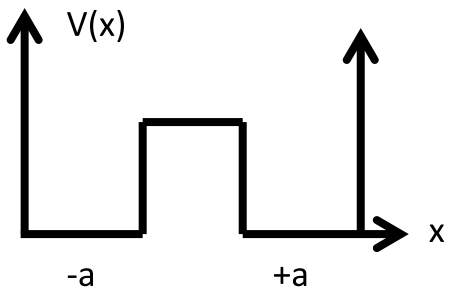
Wednesday, October 30, 2024, 18:15 – 20:15

<u>Read these instructions carefully. If you do not follow them your exam might be (partially) voided.</u>

- This exam consists of <u>3 questions in 2 pages and a formula sheet</u> at the end.
- The points for each question are indicated on the left side of the page.
- You have 2 hours to complete this exam.
- **<u>Write your name and student number on *all* answer sheets that you turn in</u>**.
- Start answering each exercise on a new page. It is ok to use front and back.
- Clearly write the total number of answer sheets that you turn in on the first page.
- Telephones, smart devices, and other electronic devices are **<u>NOT</u>** allowed.
- **<u>This is a closed book exam</u>**. Consulting reading material is **<u>not</u>** allowed.

<u>34 pts</u> **Question 1**

Assume a double well as sketched below. The ground states of each well, if isolated as with a large barrier in between, are $|\phi_L\rangle \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$ and $|\phi_R\rangle \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$. In this basis, the Hamiltonian describing the coupled well is given by the matrix elements $\langle\phi_L|\hat{H}|\phi_L\rangle = \langle\phi_R|\hat{H}|\phi_R\rangle = \varepsilon$, and $\langle\phi_R|\hat{H}|\phi_L\rangle = \langle\phi_L|\hat{H}|\phi_R\rangle = T$, where ε is a positive constant and T is a negative constant.



8 pts **a)** Explicitly write the matrix for the Hamiltonian and express the eigenstates as linear combinations of $|\phi_L\rangle$ and $|\phi_R\rangle$. What are the eigenvalues? Which eigenstate corresponds to the ground state?

8 pts **b)** What is the result of operator $\hat{P} = |\phi_L\rangle\langle\phi_R| + |\phi_R\rangle\langle\phi_L|$ on the eigenstates of the Hamiltonian? Write down the matrix corresponding to this operator in the basis of $|\phi_L\rangle$ and $|\phi_R\rangle$. Does this operator commute with the Hamiltonian? Prove that explicitly or argue why/why not.

10 pts **c)** Using a special experimental apparatus, we can make a measurement and know if it is on the left or right side of the well. This measurement is described by the operator $\hat{A} \rightarrow \begin{pmatrix}-a & 0\\0 & a\end{pmatrix}$. We have performed a measurement of $\hat{A}$ and found $-a$. What is the state of the system just after this measurement? Find how the time-dependence of the expectation value of $\hat{A}$. *Hint*: *What is* $|\phi_L\rangle$ *expressed in terms of the ground and excited state of the double well?*

8 pts **d)** Consider now that you place two electrons in the double well and let the system relax to the ground state. Use symmetry arguments to write down the total quantum state, including spin, of this two-electron system and explain why it is the correct state.

33 pts **Question 2**

For this question, consider an electron subject to the infinite square well potential:

$V(x) = \begin{cases} 0, & \text{for } \; 0 < x < a \\ \infty, & \text{otherwise} \end{cases}$, where a is a positive constant.

5 pts **a)** Solve the Schrodinger equation to show that the eigenstates and eigenenergies of this system are, respectively, $\psi_n = \sqrt{\frac{2}{a}}\sin\left({n\pi x}/{a}\right)$ and $E_n = {n^2\pi^2\hbar^2}/{2ma^2}$, with $n = 1, 2, 3, \dots$. Explicitly mention the boundary conditions you use to solve the differential equation and be clear in all the steps you take towards the solution. Do not worry about the normalization.

7 pts **b)** What are the expectation values for position and momentum for the ground state of this system? You can either do the calculation explicitly or state the final result and argue clearly why this is the answer.

6 pts **c)** Do the expectation values in calculated in (b) change in time? Here you can also either do an explicit calculation or support your statement with a clear argument.

5 pts **d)** We prepare the particle at the ground state of the well and suddenly double the well width, with the new well extending from $x = 0$ until $x = 2a$. State the new energy eigenstates and eigenenergies.

10 pts **e)** Find the time-dependent wavefunction with the initial state prepared as described in (d). *Hint*: You can express $\Psi(x, t)$ as an infinite sum, indicating how the coefficients can be calculated without the need of solving the integrals.

33 pts **Question 3**

This question is about quantum angular momentum, both orbital and spin.

8 pts **a)** Using the canonical commutation relations to show that: $[L_z, x] = i\hbar\, y$.

7 pts **b)** The spherical harmonics are eigenfunctions of the operator L_z. The eigenfunction for $l = 1$ with corresponding $m_l = 0$ is given by $Y_1^0(\theta, \phi) = \left(\frac{3}{4\pi}\right)^{1/2} \cos(\theta)$.

Using the definition for the operators $L_\pm = L_x \pm iL_y$ and the eigenfunction above find the other eigenfunctions corresponding to the same value of l.

Tip: The operators $L_{x,y,z}$ are given in the formula sheet below.

We will now discuss spin angular momentum.

5 pts **c)** Using a Stern-Gerlach apparatus you measure the state of a spin ½ particle with the axis of the apparatus at an angle θ with respect to the z axis. What are the possible values for the measurements of the spin in this axis?

The state of the "upper" branch coming from the apparatus above $|\psi\rangle$ can be written as:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle,$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of S_z with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, respectively, θ is the polar angle (angle with the z axis) and ϕ the azimuthal angle (angle in the xy plane and with respect to the x axis).

6 pts **d)** Show that the state $|\psi\rangle$ above is normalized for any angle θ or ϕ.

7 pts **e)** Assume that the Stern-Gerlach apparatus is set at angles $\theta = 60°$ and $\phi = 0°$. You take the "upper" branch of this measurement (i.e. you take $|\psi\rangle$) and do a measurement of the z component of the spin angular momentum. What are the possible outcomes and with which probabilities?

Useful formulas:

Schrodinger equation $i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$

Time-independent Schrodinger equation $H\psi = E\psi \qquad \Psi = \psi e^{-iEt/\hbar}$

Hamiltonian operator $H = -\frac{\hbar^2}{2m}\nabla^2 + V$

Momentum operator $p = -i\hbar\,\nabla$

De Broglie wavelength $\lambda = h/p$

Time-dependence of expectation value $\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle [H,Q]\rangle + \langle \frac{\partial Q}{\partial t}\rangle$

Generalized uncertainty principle $\sigma_A\sigma_B \geq \left|\frac{1}{2i}\langle [A,B]\rangle\right|$

Heisenberg Uncertainty principle $\sigma_x\sigma_p \geq \hbar/2$

Canonical commutator $[x,p] = i\hbar$

Angular momentum $\vec{L} = \vec{r} \times \vec{p}$

$$[L_x, L_y] = i\hbar L_z \;;\; [L_y, L_z] = i\hbar L_x \;;\; [L_z, L_x] = i\hbar L_y$$

In spherical coordinates:

$$L_z = -i\hbar\ {\partial}/{\partial\phi}$$

$$L_x = -i\hbar\left(-\sin\phi\ {\partial}/{\partial\theta} - \cos\phi\cot\theta\ {\partial}/{\partial\phi}\right)$$

$$L_y = -i\hbar\left(+\cos\phi\ {\partial}/{\partial\theta} - \sin\phi\cot\theta\ {\partial}/{\partial\phi}\right)$$

Pauli matrices $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \;;\; \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \;;\; \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Trigonometric relations

$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

Useful integrals

$$\int x\sin^2(ax)\,dx = \left(\frac{1}{8a^2}\right)\{-2ax[\sin(2ax) - ax] + \cos(2ax)\}$$

$$\int \sin(ax)\cos(ax)\,dx = -\frac{\cos^2(ax)}{2a}$$