

Resit Exam Quantum Physics 1 – 2024/2025

Friday, January 31, 2025, 18:15 – 20:15

Read these instructions carefully. If you do not follow them your exam might be (partially) voided.

- This exam consists of 3 questions in 2 pages and a formula sheet at the end.
- The points for each question are indicated on the left side of the page.
- You have 2 hours to complete this exam.
- **Write your name and student number on all answer sheets that you turn in.**
- Start answering each exercise on a new page. It is ok to use front and back.
- **Clearly write the total number of answer sheets that you turn in on the first page.**
- Telephones, smart devices, and other electronic devices are **NOT** allowed.
- **This is a closed book exam.** Consulting reading material is **not** allowed.

34 pts Question 1

A simple model for a quantum dot is the three-dimensional infinite potential well, for which the potential is given below:

$$V = \begin{cases} 0, & \text{for } 0 < x, y, z < d \\ \infty, & \text{otherwise} \end{cases},$$

where d is the lateral size of the quantum dot, and x , y , and z are Cartesian coordinates.

13 pts **a)** Show that the energy eigenstates of this simple model is given by:

$$\psi(x, y, z) = \sqrt{\left(\frac{2}{d}\right)^3} \sin\left(\frac{n\pi x}{d}\right) \sin\left(\frac{m\pi y}{d}\right) \sin\left(\frac{l\pi z}{d}\right),$$

where n , m and l are integers – i.e. $n, m, l = 1, 2, 3, \dots$

You **do not need to solve the integrals** for the normalization constant, but **DO indicate** explicitly the integrals which have to be solved and what they should equal to.

7 pts **b)** Call the distinct energies $E_1, E_2, E_3, \dots$, in order of increasing energy. Find E_1, E_2, E_3 and E_4 . Determine their degeneracies (that is, the number of different states that share the same energy).

7 pts **c)** Suppose you have one of these quantum dots and charge it with two electrons. Write down the possible combinations for their joint spin states (spinors) which are symmetric and antisymmetric with respect to particle exchange.

7 pts **d)** Assume that we prepare the spatial component in the state: $\psi_{111}^a \psi_{211}^b - \psi_{211}^a \psi_{111}^b$, where ψ_{nml}^i represents the wavefunction of electron $i = a, b$, with quantum numbers n, m, l . What is spinor of the system if the electrons are prepared in the state above? You can answer using symmetry arguments or explicitly writing down the spinor.

33 pts Question 2

An operator $\hat{A}$, which represents the observable A , has two normalized eigenstates, $|\psi_1\rangle$ and $|\psi_2\rangle$, with the respective eigenvalues a_1 and a_2 .

7 pts

a) What is the matrix which represents $\hat{A}$ in its basis?

6 pts

b) We perform a measurement of the observable A and obtain the value a_1 . What is the state of the system immediately after this measurement?

Now assume another operator $\hat{B}$, which represents the observable B . It has two normalized eigenstates, $|\phi_1\rangle$ and $|\phi_2\rangle$, with the respective eigenvalues b_1 and b_2 . These states can be written in as a function of the eigenvectors of $\hat{A}$ as:

$$|\phi_1\rangle = \frac{3}{5}|\psi_1\rangle + \frac{4}{5}|\psi_2\rangle \quad \text{and} \quad |\phi_2\rangle = \frac{4}{5}|\psi_1\rangle - \frac{3}{5}|\psi_2\rangle.$$

12 pts

c) What is the matrix representing $\hat{B}$ in the basis of $\hat{A}$?

Tip: Start by showing that $|\psi_1\rangle = \frac{3}{5}|\phi_1\rangle + \frac{4}{5}|\phi_2\rangle$ and $|\psi_2\rangle = \frac{4}{5}|\phi_1\rangle - \frac{3}{5}|\phi_2\rangle$ and then calculate the matrix elements.

8 pts

d) If we prepare the system in the state $|\psi_2\rangle$ and perform a measurement of the observable B , what are the possible outcomes of the measurement? With what probabilities?

33 pts Question 3

Consider a particle of mass m in the one-dimensional harmonic oscillator. The Hamiltonian is:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

where ω is the angular frequency.

It is useful to know the ladder operators and their commutator, given by

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \quad \text{and} \quad [a_-, a_+] = 1.$$

8 pts

a) Show that the Hamiltonian can be written as $H = \hbar\omega(a_{\pm}a_{\mp} \pm 1/2)$. You can derive it for only one of the choices of sign (top or bottom choice).

8 pts

b) Use the ladder operators to show that the ground state wavefunction is given by:

$$\psi_0 = Ae^{-\frac{m\omega}{2\hbar}x^2}$$

10 pts

c) Suppose we initialize a particle in the harmonic oscillator at time $t = 0$ in the state:

$\psi(x, t = 0) = \sqrt{\frac{2}{3}}\psi_0(x) + \sqrt{\frac{1}{3}}e^{i\pi/3}\psi_1(x)$. What is the expectation value for the energy for the particle at this state? What are the possible values for the measurement of the energy?
Good to know: $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$ and $a_-\psi_n = \sqrt{n}\psi_{n-1}$.

7 pts

d) Given the initial state above, write down the time-evolution of the wavefunction, i.e. $\psi(x, t)$.

Useful formulas:

Schrodinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

Time-independent Schrodinger equation

$$H\psi = E\psi \quad \Psi = \psi e^{-iEt/\hbar}$$

Hamiltonian operator

$$H = -\frac{\hbar^2}{2m} \nabla^2 + V$$

Momentum operator

$$p = -i\hbar \nabla$$

De Broglie wavelength

$$\lambda = h/p$$

Time-dependence of expectation value

$$\frac{d\langle Q \rangle}{dt} = \frac{i}{\hbar} \langle [H, Q] \rangle + \langle \frac{\partial Q}{\partial t} \rangle$$

Generalized uncertainty principle

$$\sigma_A \sigma_B \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|$$

Heisenberg Uncertainty principle

$$\sigma_x \sigma_p \geq \hbar/2$$

Canonical commutator

$$[x, p] = i\hbar$$

Angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$[L_x, L_y] = i\hbar L_z ; [L_y, L_z] = i\hbar L_x ; [L_z, L_x] = i\hbar L_y$$

In spherical coordinates:

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$L_x = -i\hbar \left(-\sin \phi \frac{\partial}{\partial \theta} - \cos \phi \cot \theta \frac{\partial}{\partial \phi} \right)$$

$$L_y = -i\hbar \left(+\cos \phi \frac{\partial}{\partial \theta} - \sin \phi \cot \theta \frac{\partial}{\partial \phi} \right)$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Trigonometric relations

$$\sin(a \pm b) = \sin(a) \cos(b) \pm \cos(a) \sin(b)$$

$$\cos(a \pm b) = \cos(a) \cos(b) \mp \sin(a) \sin(b)$$

Useful integrals

$$\int \sin^2(ax) dx = \frac{x}{2} - \frac{\sin(ax) \cos(ax)}{2a} + \text{const.}$$

$$\int x \sin^2(ax) dx = \left(\frac{1}{8a^2} \right) \{ -2ax[\sin(2ax) - ax] + \cos(2ax) \}$$