1 a) $H\Psi = \epsilon\Psi \Rightarrow \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x,y,z)\right)\Psi = \epsilon\Psi$

Separation of variables: $\Psi(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] XYZ = \epsilon XYZ \qquad \text{for } 0 \leq x,y,z \leq d$$

$$-\frac{\hbar^2}{2m}\left[YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2}\right] = \epsilon XYZ$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right] = \epsilon$$

Solving for X:

$$-\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \text{constant} \equiv \epsilon_x \Rightarrow \frac{\partial^2 X}{\partial x^2} = -k_x^2 X$$

$$k_x \equiv \frac{\sqrt{2m\epsilon_x}}{\hbar}$$

Solutions:

$$X(x) = A'\sin(k_x x) + B'\cos(k_x x)$$

$X(x=0) = 0 \Rightarrow B' = 0$

$X(x=d) = 0 \Rightarrow A'\sin(k_x d) = 0 \Rightarrow k_x d = n\pi \Rightarrow k_x = n\pi/d$

So: $X = A'\sin\left(\frac{n\pi x}{d}\right)$ and similar for Y and Z.

$$\Rightarrow \Psi(x,y,z) = A\sin\left(\frac{n\pi x}{d}\right)\sin\left(\frac{m\pi y}{d}\right)\sin\left(\frac{\ell\pi z}{d}\right)$$

To find the constant A (normalization):

$$\iiint|\Psi|^2 d^3r = 1 \Rightarrow A^2\int_0^d \sin^2\left(\frac{n\pi x}{d}\right)dx\int_0^d \sin^2\left(\frac{m\pi y}{d}\right)dy\int_0^d \sin^2\left(\frac{\ell\pi z}{d}\right)dz = 1$$

b) $\epsilon_T = \epsilon_x + \epsilon_y + \epsilon_z = \frac{\hbar^2 \pi^2}{2md^2}(n^2 + m^2 + l^2)$ $\quad n, m, l = 1, 2, 3, 4 \ldots$

$\epsilon_1 \Rightarrow n = m = l = 1 \Rightarrow \epsilon = \frac{3}{2} \frac{\hbar^2 \pi^2}{2md^2}$ degeneracy 1

ϵ_2

n	m	l
2	1	1
1	2	1
1	1	2

$\epsilon_2 = \frac{6}{2} \frac{\hbar^2 \pi^2}{2md^2} \Rightarrow \epsilon_2 = 3 \frac{\hbar^2 \pi^2}{2md^2}$

deg. 3

ϵ_3

n	m	l
2	2	1
2	1	2
1	2	2

$\epsilon_3 = \frac{9}{2} \frac{\hbar^2 \pi^2}{2md^2}$

deg. 3

ϵ_4 $\quad n = m = l = 2$ $\quad \epsilon_4 = \frac{12}{2} \frac{\hbar^2 \pi^2}{2md^2} \Rightarrow \epsilon_4 = 6 \frac{\hbar^2 \pi^2}{2md^2}$

deg. 1

c) Possible spin states:

$$|s=0\ m_s=0\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]$$ singlet (antisymmetric)

$$\left.\begin{aligned} |1\ 1\rangle &= |\uparrow\uparrow\rangle \\ |1\ 0\rangle &= \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right] \\ |1\ {-1}\rangle &= |\downarrow\downarrow\rangle \end{aligned}\right\} \text{triplet (symmetric)}$$

---

d) $\Psi_{a,b} = \psi^a_{111}\psi^b_{211} - \psi^a_{211}\psi^b_{111}$

Upon particle exchange: $a \to b \quad b \to a$

$$\Psi_{b,a} = \psi^b_{111}\psi^a_{211} - \psi^b_{211}\psi^a_{111} = -\Psi_{a,b}$$

Antisymmetric.

Therefore, spin state should be one of the triplet states.

2) a) $\hat{A} \longrightarrow \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$

---

b) The system collapses to state $|\psi_1\rangle$.

---

c) $|\phi_1\rangle = \frac{3}{5}|\psi_1\rangle + \frac{4}{5}|\psi_2\rangle \qquad |\phi_2\rangle = \frac{4}{5}|\psi_1\rangle - \frac{3}{5}|\psi_2\rangle$

Matrix elements given by: $\langle \psi_{1,2}|\hat{B}|\psi_{1,2}\rangle$

$$3|\phi_1\rangle + 4|\phi_2\rangle = \frac{9}{5}|\psi_1\rangle + \frac{12}{5}|\psi_2\rangle + \frac{16}{5}|\psi_1\rangle - \frac{12}{5}|\psi_2\rangle$$

$$3|\phi_1\rangle + 4|\phi_2\rangle = \frac{25}{5}|\psi_1\rangle \Rightarrow |\psi_1\rangle = \frac{3}{5}|\phi_1\rangle + \frac{4}{5}|\phi_2\rangle$$

$$4|\phi_1\rangle - 3|\phi_2\rangle = \frac{25}{5}|\psi_2\rangle \Rightarrow |\psi_2\rangle = \frac{4}{5}|\phi_1\rangle - \frac{3}{5}|\phi_2\rangle$$

$$\langle\psi_1|\hat{B}|\psi_1\rangle = \frac{1}{25}\left[3\langle\phi_1| + 4\langle\phi_2|\right]\hat{B}\left[3|\phi_1\rangle + 4|\phi_2\rangle\right] = \frac{9}{25}b_1 + \frac{16}{25}b_2$$

$$\langle\psi_1|\hat{B}|\psi_2\rangle = \frac{1}{25}\left[3\langle\phi_1| + 4\langle\phi_2|\right]\hat{B}\left[4|\phi_1\rangle - 3|\phi_2\rangle\right] = \frac{12}{25}b_1 - \frac{12}{25}b_2$$

$$\langle\psi_2|\hat{B}|\psi_1\rangle = \frac{1}{25}\left[4\langle\phi_1| - 3\langle\phi_2|\right]\hat{B}\left[3|\phi_1\rangle + 4|\phi_2\rangle\right] = \frac{12}{25}b_1 - \frac{12}{25}b_2$$

$$\langle\psi_2|\hat{B}|\psi_2\rangle = \frac{1}{25}\left[4\langle\phi_1| - 3\langle\phi_2|\right]\hat{B}\left[4|\phi_1\rangle - 3|\phi_2\rangle\right] = \frac{16}{25}b_1 + \frac{9}{25}b_2$$

$$\hat{B} \longrightarrow \frac{1}{25}\begin{array}{c c} & \begin{array}{cc} |\psi_1\rangle & |\psi_2\rangle \end{array} \\ \begin{array}{c} \langle\psi_1| \\ \langle\psi_2| \end{array} & \begin{pmatrix} 9b_1 + 16b_2 & 12b_1 - 12b_2 \\ 12b_1 - 12b_2 & 16b_1 + 9b_2 \end{pmatrix} \end{array}$$

---

d) $|\psi_2\rangle = \frac{4}{5}|\phi_1\rangle - \frac{3}{5}|\phi_2\rangle \Rightarrow$ b_1 with prob $\frac{16}{25}$

b_2 with prob $9/25$

3 a) $a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp ip + m\omega x\right)$

$$a_+ + a_- = \frac{2}{\sqrt{2\hbar m\omega}}\, m\omega x$$

$$a_- - a_+ = \frac{2}{\sqrt{2\hbar m\omega}}\, ip$$

$$\Rightarrow x = \sqrt{\frac{\hbar}{2m\omega}}\left(a_+ + a_-\right)$$

Important! $a_+a_- \neq a_-a_+$

$$\Rightarrow p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a_+ - a_-\right)$$

$$x^2 = \left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 (a_+ + a_-)(a_+ + a_-) = \left(\frac{\hbar}{2m\omega}\right)\left[a_+^2 + a_+a_- + a_-a_+ + a_-^2\right]$$

$$p^2 = -\left(\frac{\hbar m\omega}{2}\right)\left[a_+^2 - a_+a_- - a_-a_+ + a_-^2\right]$$

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

$$= -\frac{\hbar\omega}{4}\left[a_+^2 - a_-a_+ - a_+a_- + a_-^2\right] + \frac{\hbar\omega}{4}\left[a_+^2 + a_+a_- + a_-a_+ + a_-^2\right]$$

$$= \frac{\hbar\omega}{2}\left[a_+a_- + a_-a_+\right]$$

Using $[a_-, a_+] = 1 \Rightarrow a_-a_+ - a_+a_- = 1$

$\Rightarrow a_-a_+ = a_+a_- + 1$

$$= \frac{\hbar\omega}{2}\left[a_+a_- + a_+a_- + 1\right]$$

$$\Rightarrow \hat{H} = \hbar\omega\left(a_+a_- + \tfrac{1}{2}\right)$$

b) $a_- \Psi_0 = 0$

$$\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\Psi_0 = 0$$

$$\hbar\frac{d\Psi_0}{dx} + m\omega x\Psi_0 = 0 \Rightarrow \frac{d\Psi_0}{dx} = -\frac{m\omega x}{\hbar}\Psi_0$$

$$\int\frac{d\Psi_0}{\Psi_0} = -\frac{m\omega}{\hbar}\int x\,dx \Rightarrow \ln\Psi_0 = -\frac{m\omega}{\hbar}\frac{x^2}{2} + c$$

$$\Psi_0 = A e^{-\frac{m\omega}{2\hbar}x^2}$$

---

c) $\langle H\rangle = ?$ $\qquad |\Psi\rangle = \sqrt{\frac{2}{3}}|\Psi_0\rangle + \sqrt{\frac{1}{3}}e^{i\pi/3}|\Psi_1\rangle$

$\langle H\rangle = \langle\Psi|\hat{H}|\Psi\rangle$ or $\int\Psi^*\hat{H}\Psi\,dx$ (dif. notation, same results)

$$= \left(\sqrt{\frac{2}{3}}\langle\Psi_0| + \sqrt{\frac{1}{3}}e^{-i\pi/3}\langle\Psi_1|\right)\left(\hbar\omega(a_+a_- + \tfrac{1}{2})\right)\left(\sqrt{\frac{2}{3}}|\Psi_0\rangle + \sqrt{\frac{1}{3}}e^{i\frac{\pi}{3}}|\Psi_1\rangle\right)$$

$a_+a_-|\Psi_0\rangle = 0$ $\left(\text{since } a_-|\Psi_0\rangle = 0\right)$

$a_+a_-|\Psi_1\rangle = a_+\left(\sqrt{1}|\Psi_0\rangle\right) = \sqrt{0+1}\,|\Psi_1\rangle = |\Psi_1\rangle$

So: $\langle H\rangle = \left(\sqrt{\frac{2}{3}}\langle\Psi_0| + \sqrt{\frac{1}{3}}e^{-i\pi/3}\langle\Psi_1|\right)(\hbar\omega)\left[\frac{1}{2}\sqrt{\frac{2}{3}}|\Psi_0\rangle + \sqrt{\frac{1}{3}}e^{i\pi/3}\left(|\Psi_1\rangle + \frac{1}{2}|\Psi_1\rangle\right)\right]$

$$= \hbar\omega\left(\frac{1}{2}\frac{2}{3}\langle\Psi_0|\Psi_0\rangle + \frac{1}{3}\frac{3}{2}\langle\Psi_1|\Psi_1\rangle\right) = \hbar\omega\left(\frac{1}{3} + \frac{1}{2}\right)$$

$$\Rightarrow \boxed{\langle H\rangle = \frac{5}{6}\hbar\omega}$$

d) t-ev. operator $\hat{U} = e^{\frac{i\hat{H}t}{\hbar}}$

$$\Psi(x,t=0) = \sqrt{\frac{2}{3}}\,\psi_0 + \sqrt{\frac{1}{3}}\,e^{+i\pi/3}\,\psi_1$$

$$\epsilon_n = \hbar\omega(n+1/2) \qquad n = 0, 1, 2, \dots$$

$$\epsilon_0 = \hbar\omega/2 \qquad \epsilon_1 = \frac{3}{2}\hbar\omega$$

$$\Psi(x,t) = \sqrt{\frac{2}{3}}\,e^{i\epsilon_0 t/\hbar}\,\psi_0 + \sqrt{\frac{1}{3}}\,e^{i\pi/3}\,e^{i\epsilon_1 t/\hbar}\,\psi_1$$

$$= \sqrt{\frac{2}{3}}\,e^{\frac{i\hbar\omega t}{2\hbar}}\,\psi_0 + \sqrt{\frac{1}{3}}\,e^{i(\pi/3 + \frac{3}{2}\hbar\omega t/\hbar)}\,\psi_1$$

$$\boxed{\Psi(x,t) = \sqrt{\frac{2}{3}}\,e^{i\frac{\omega}{2}t}\,\psi_0 + \sqrt{\frac{1}{3}}\,e^{i(\frac{3\omega}{2}t + \pi/3)}\,\psi_1}$$