Name: Student Number:

Mechanics and Relativity: R1

September 27, 2023, Aletta Jacobshal

Duration: 90 mins

Before you start, read the following:

- There are 2 problems, for a total of 23 points.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

	Points
Problem 1:	8
Problem 2:	15
Total:	23
GRADE (1 + # Total/(9/23))	

Useful equations:

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$\Delta t \geq \Delta s \geq \Delta \tau$$

Possibly relevant equations:

$$F = G\frac{Mm}{r^2}; \qquad F = ma; \qquad PV \propto k_b T; \qquad F = \frac{dp}{dt}$$

Possibly relevant numbers:

$$c = 299792458 \text{ m/s} \tag{1}$$

Approximation (if $v \ll c$) (known as Binomial expansion – see lecture clip)

$$(1 - (v/c)^2)^{1/2} \simeq 1 - \frac{1}{2}(v/c)^2 \tag{2}$$

Question 1: Conceptual Warm-up (8 pts)
Answer the following question, clearly and concisely. For some of the problems you might want to sketch the problem if that helps. None the questions require you to derive something, but rather conceptually address the question.

(a) **(1 pt)** Provide a proper definition of an event.

(b) **(2 pts)** In a Galilean Universe do the coordinates of an event change if two observers move with a relative velocity v_x (i.e. if the two observers are in different inertial frames), where v_x the speed in the x direction? Write down the coordinates of an event for these two observers assuming that their clocks are synchronized.

(c) **(1 pt)** Describe (words) how coordinates change in special relativity (SR). How does that differ from Galilean Relativity?

(d) **(3 pts)** Assuming SR, suppose an observer measures some velocity v_x of an object in the observers rest frame moving in the $+x$ direction. Another observer is moving with constant speed in the $+y$ direction. Will the speed in the $+x$, as observed by the observer moving in the $+y$ direction change? Why or why not? How does this compare to a Galilean Universe? *Hint: use the fact that for constant velocities, we can write $v = \Delta x/\Delta t$, where the intervals can be considered the measurement of the coordinates of the moving object between two events.*

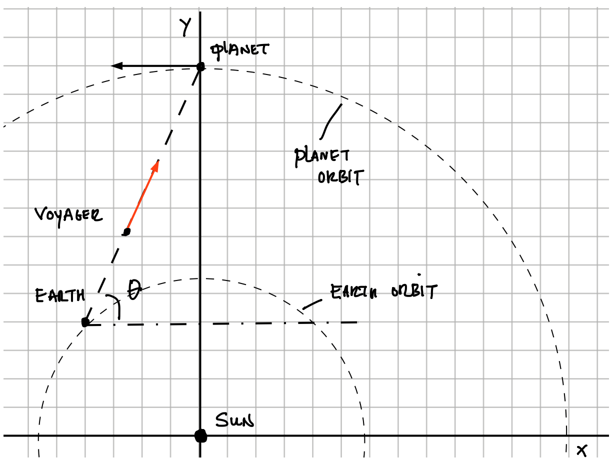
(e) **(1 pt)** While both space and time coordinates change in special relativity, the spacetime interval (Δs^2) and proper time are frame independent as we learned in the lecture (note that we could also state that the magnitude $|\Delta s|$ is frame independent). We have also explained that because of the 'coupling' between time and space in SR, the frame independent quantities should both rely on space and time because these are both frame dependent. An infinitesimal displacement can be written as a 4-vector $ds \equiv (dt, dx, dy, dz)$. At the same time, we can define an infinitesimal interval in proper time as $d\tau$. Define a velocity-like (which in SI Units would have the dimension of velocity) quantity which 'magnitude' could be frame independent. Argue why this object could be frame independent.

Question 2: Voyager (15 pts)

The Voyager I and II are two monumental discovery spacecraft that have been the first to reach the outer planets of our solar system. Both launched in 1977, they are currently the only two human-made objects to have ever left our solar system. In order for anything to leave our solar system, requires to accelerate beyond the systems escape velocity. To reach these high velocities, many spacecraft use a (so-called) gravitational slingshot (Kondratyuk, 1938), by which the gravitational potential of, here a planet, is used to change the direction of motion of the spacecraft, **but not** its speed (= magnitude of the velocity) w.r.t. the planet (in the vacuum of space there is no friction). The Voyagers both used gravitational slingshots by Jupiter and Uranus to accelerate beyond the solar-systems escape velocity.

(a) **(1 pt)** Imagine a spacecraft approaching a planet and undergo a slingshot. Draw/sketch a space-time diagram showing such a gravitational slingshot in a frame attached to the planet used to perform the slingshot. As your spatial axis use r', the radial distance from the planet. Label the spacetime coordinates the spacecraft changes course as event A. Explain your drawing. You can neglect the acceleration of the planet and assume its motion can be described by an inertial frame. You do not have to use SR units (so time and space can have different scales and the speed of light is not necessarily making a 45 degree angle). You can use the box below or the drawing paper at the back of the exam.

(b) **(1 pt)** Next sketch a spacetime diagram attached to the Sun (a frame where the Sun is at rest). Choose as your horizontal axis the radial direction r as observed from the Sun. Draw the worldline of the spacecraft as it passes the planet at distance r_{planet} from the Sun. Explain why, as measured in this frame, the spacecraft changes velocity (in the radial direction). Label the space-time coordinates of the gravitational slingshot as event A. Again you can neglect any acceleration of the Sun and assume it is in an inertial frame. Furthermore, you can assume the planet's motion is non-relativistic w.r.t. the motion of the Sun, which means velocities in these frames (planet vs Sun) are related by Galilean transformations.



In a *Galilean universe*, we can forget about the time axis. We draw a frame at rest with respect to the Sun. The spacecraft will pass around the back of the planet at $x = 0$. The planet is moving with velocity U in the $-x$ direction.

(c) **(2 pts)** The Voyager is moving away from Earth towards the planet with constant velocity $\vec{v}^{\text{in}}$. Consider a frame co-moving with the planet. Label the coordinates in this frame with $'$. In this frame, which coordinates have changed compared the frame at rest (as shown in the figure)? What are the components of the velocity v^{in} and after v^{out} parallel to the x and y

axis in this new frame assuming the slingshot changes the direction of the Voyager by 2θ (the trajectory is 'mirrored' in the x axis)?

(d) **(2 pts)** Next, write down the components of the velocity in the frame at rest w.r.t. the Sun.

(e) **(2 pts)** Show that the outgoing speed, as observed in the rest from of the Sun, is given by

$$|\vec{v}|^{\text{out}} = (|\vec{v}|^{\text{in}} + 2U)\sqrt{1 - \frac{4U|\vec{v}|^{\text{in}}(1-\cos\Theta)}{(|\vec{v}|^{\text{in}} + 2U)^2}}. \tag{3}$$

(f) **(2 pts)** Suppose the planet is moving with the same speed as the spacecraft, i.e. $|\vec{v}|^{\text{in}} = U$, simplify the expression derived in the previous question. What is the outgoing speed in the limit $\Theta \to 0$? What about when the incoming spacecraft is only moving in the $+y$ direction?

In an *Einstein universe*, we would have to consider the time coordinate. Let us check how good our assumption of a Galilean Universe is (for this problem), by comparing clocks in a reference frame on Earth and a reference frame attached to the spacecraft. Let us neglect the motion of the Earth around the Sun (in the language of Moore, this is our HOME frame).

(g) **(1 pt)** Let us now suppose the spacecraft has reached a speed v^{final} (after two slingshots) by the time it exits our solar system. The diameter of our solar system is about 240 AU. 1AU is the distance between the Earth and the Sun, and is about 500 light seconds (which you could have guessed by noting that it takes about 8 min for light to travel from the Sun to our surface). The Voyager I 'left' our solar system in 2012. Based on this what has been the average (radial) speed (in units of the speed of light) of the Voyager I?

(h) **(2 pts)** Assuming the Voyager I is traveling with this constant speed, what is the *difference* between the time registered on the spacecraft (today) and a a clock on Earth (today), measured in seconds?

(i) **(2 pts)** In reality the speed of the Voyager I currently is a little higher, with $v_r^{\text{current}} \simeq 5.7 \times 10^{-5}$ (as measured on Earth). The nearest star (Proxima Centauri) is still about 4.2 ly away. Again, estimate the time difference between a clock on Earth and on the Voyager once it has reached Proxima Centauri. Express your answer in hours.

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