

Mechanics and Relativity: R1

September 26, 2024, Exam Hall 4, Aletta Jacobshal

Duration: 90 mins

Before you start, read the following:

- There are 2 problems, for a total of 30 points.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

	Points
Problem 1:	10
Problem 2:	20
Total:	30
GRADE (1 + 3 # Total /10)	

Useful equations:

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

$$\Delta t \geq \Delta s \geq \Delta \tau$$

Time dilation:

$$\Delta t' = \gamma^{-1} \Delta t \quad (1)$$

with

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad (2)$$

if β is velocity in SR units. Possibly relevant numbers:

$$c = 299792458 \text{ m/s} \quad (3)$$

Question 1: Conceptual Warm-up (10 pts)

Assume $c = 1$ (SR units). Answer the following question, clearly and concisely. For some of the problems, you might want to sketch the problem if that helps. None the questions require you to derive something, but rather conceptually address the question.

- (a) **(3 pts)** Which worldlines in Fig. 1 (A to F) present an observer that is i) at rest, ii) moving at constant velocity iii) accelerating or decelerating iv) a combination of these or v) non-physical. Explain your answer.

- A: The first worldline belongs to an observer at rest. **(1/2 pts)**
- B: An observer that accelerates and ends at rest. However, it accelerates beyond the speed of light (<45 degrees) which is non-physical. **(1/2 pts)**
- C: This observer is moving at constant velocity. After infinite deceleration, it comes to rest. **(1/2 pts)**
- D: This worldline presents an observer moving at a constant velocity, then decelerating and accelerating to move with constant velocity in another direction. However, the velocity of this observer in both segments is larger than the speed of light, i.e. non-physical. **(1/2 pts)**
- E: This is the worldline of light, which moves at constant velocity c . **(1/2 pts)**
- F: This worldline presents an observer that is accelerating, however, it is accelerating beyond the speed of light and hence non-physical. **(1/2 pts)**.

- (b) **(2 pts)** Consider two observers, Ando and Amy. Assume Amy is located at the origin. Ando is moving away from the origin with constant velocity $\vec{v}$. Do the coordinates (t, x, y, z) of an event as observed by Amy differ from those observed by Ando in a Galilean universe? Write down the coordinates of this event as observed by Ando assuming their clocks are synchronized.

Yes, coordinates do change **(2/3 pt)**. In a Galilean universe, time is absolute. Amy is measuring coordinates (t, x, y, z) and Ando is moving away with velocity $\vec{v} = (v_x, v_y, v_z)$ **(2/3 pt)** Ando will measure coordinates of the same event as $(t, x - v_x t, y - v_y t, z - v_z t)$ **(2/3 pt)**

Name:

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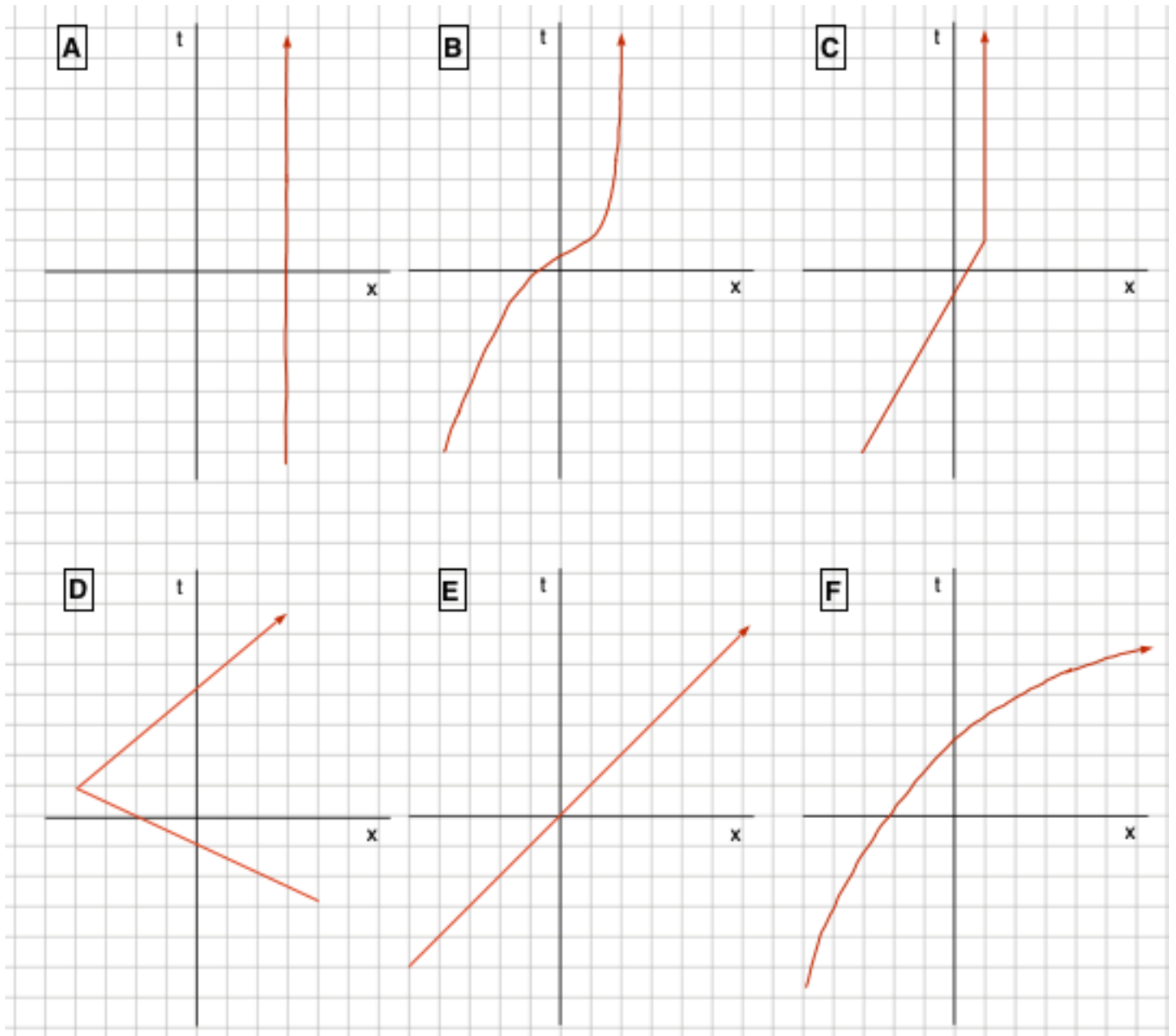


Figure 1: Worldlines. We use SR units with each checker box represents 1 ls on both axis.

- (c) **(1 pt)** Describe how this answer would change if Ando had a constant speed (as opposed to a constant velocity).

If Ando had a constant speed, then the velocity itself does not need to be constant **(1/2 pt)**. For example, if Ando is rotating in the $x - y$ plane, e.g. $\vec{v}(t) = (v \cos t, v \sin t, 0)$, then $|\vec{v}(t)| = v$, but Ando would be accelerating at all times. Galilean transformations only apply to inertial frames **(1/2 pt)**, i.e. those that move with constant velocity. Ando's would not be an inertial frame.

- (d) **(2 pts)** How does acceleration transform under a Galilean transformation? What does this tell you about Newton's second law of motion?

Acceleration is defined as $\vec{a} = d\vec{v}/dt$ **(1/2 pt)**. In a Galilean universe, if two inertial frames are moving with relative and constant velocity $\vec{V}$, $\vec{v}' = \vec{v} - \vec{V}$ **(1/2 pt)**. Then $\vec{a}' = \frac{d(\vec{v} - \vec{V})}{dt} = \frac{d\vec{v}}{dt} = \vec{a}$ since $\vec{V}$ is constant by definition **(1/2 pt)**. Newton's second law states that $\vec{F} = m\vec{a}$, so this implies Newton's law is invariant under Galilean transformations **(1/2 pt)**.

- (e) **(1 pt)** Contrary to Galilean relativity, in Special Relativity (SR), time is no longer absolute. In the former, spatial coordinates and velocity components only transform when we consider a frame that is moving in the direction of that component, e.g. for a frame moving in the x direction only, V_x we have $x' = x - V_x t$ and $v'_x = v_x - V_x$, while $y' = y$, $z' = z$ and similarly for the velocities. Do you expect this to be true also for velocity components in SR? Why or why not?

For example, consider v_x and a frame moving with velocity β_y . Since $v_x \equiv dx/dt$, even if there is no coordinate transformation in the x direction (i.e. $\beta_x = 0$), since coordinate time is frame dependent **(1/2 pt)**, the velocity components will all transform, independent of the direction of the velocity of the other frame **(1/2 pt)**. – another answer: these transformations imply no speed limit so velocity can not be transformed in this way in special relativity.

- (f) **(1 pt)** What about acceleration in SR? Will this be (inertial) frame invariant? And what does this imply for Newton's second law in SR? Assume that the transformations of coordinate time and velocity do not cancel.

Similarly, consider a_x and a frame moving with velocity β_y . Since $a_x \equiv dv_x/dt$ and we just argued that v_x transforms because t is frame dependent **(1/2 pt)**, $a'_x \neq a_x$ for inertial frames with relative velocity β_y . This means that in SR, Newton's laws of motion are not frame independent (and hence they do not describe forces correctly in an SR universe) **(1/2 pt)**.

Question 2: Twin paradox revisited (20 pts)

Consider triplets, Fang, Julia and Alina, who each carry a clock. Assume Fang is on Earth. In Fang's frame, Julia passes by with speed $|v| = v$ to the right (in an $x - t$ space-time diagram). As she passes, Julia and Fang synchronize their clocks to $t = t' = 0$, where we label Fang's frame with t, x and Julia's frame with primed coordinates. In the same frame, their sister Alina is traveling towards Fang from some distant planet. She also is traveling with speed $|v| = v$, coming from the right (moving to the left). On her way back to Fang, she passes by her sister Julia. As they pass, Alina sets her clock to Julia's clock, $t'' = t'$ (not the other way around!). We want to determine the relation between the time on Fang's clock and the time on Alina's clock once Alina arrives on Earth. We want to know if this relation is frame-independent. The frames we will consider are either comoving with Fang, with Julia or with Alina. $c = 1$ in this question.

- (a) **(2 pts)** Do we expect the relation between the two times to be the same as determined in each frame? Why or why not?

Yes, the answer should be the same **(1 pt)**. While coordinate time is frame dependent, the relation between coordinate times as measured by observers in relative motion in different inertial frames should not change. If it would change, we could imagine boosting to a frame where the two times would be equal, which can not be true due to the fact that we know coordinate time IS frame dependent for observers in relative motion **(1 pt)**.

- (b) **(6 pts)** Use (t, x) , (t', x') , and (t'', x'') for Fang, Julia and Alina's frame respectively. **Sketch** the 3 space-time diagrams comoving with i) Fang, ii) Julia and iii) Alina. Draw the worldlines of the three sisters in each diagram. Label these. In Fang's frame, mark the coordinate time interval between Julia passing Fang and Julia passing Alina as Δt_1 and the interval between Julia passing Alina and Alina reaching Fang on Earth with Δt_2 . Do the same in the two other frames, now with the appropriate primes for the coordinate time. You can use the checkered paper at the back of the exam.

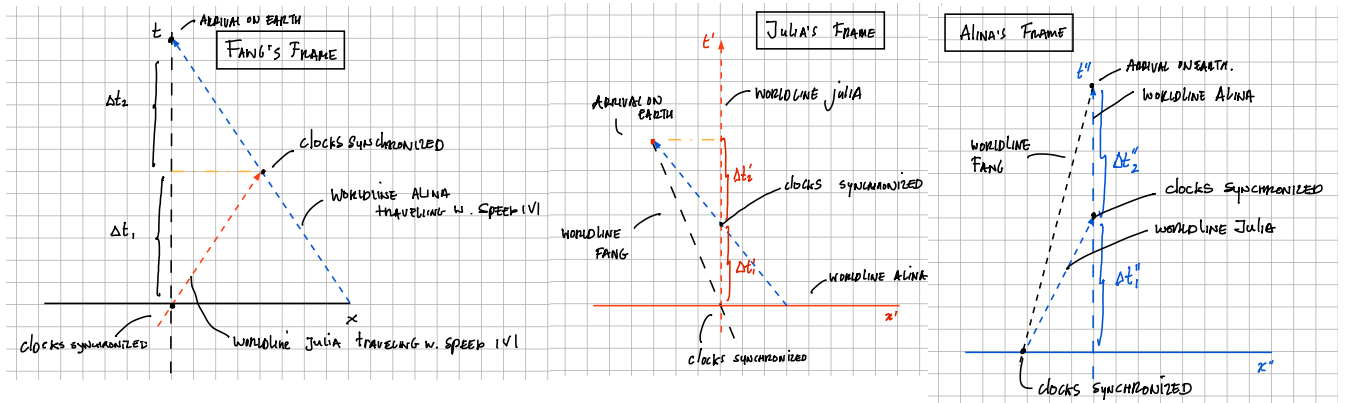


Figure 2: 2 pts for each correct diagram. Labelling both axis (1/2 pt), Labelling all 3 worldlines (1 pt), and labelling the two intervals (1/2 pt).

- (c) (4 pts) Starting in Fang's frame, what is the relation between the time interval Δt_1 and the time interval measured by Julia, i.e. $\Delta t'_1$? What about Δt_2 and $\Delta t'_2$? How are Δt_1 and Δt_2 related? What does this imply for the relation between the time measured on Alina's clock $\Delta t''_1 + \Delta t''_2 = T_{\text{Alina}}$ and Fang's clock $\Delta t_1 + \Delta t_2 = T_{\text{Fang}}$? (recall that both these times are the times measured on the clocks comoving with either Fang or Alina). Does the result make sense?

In Fang's frame, Julia's clock should run slower: $\Delta t'_1 = \sqrt{1 - |v|^2} \Delta t_1 = \Delta t_1 / \gamma$ (1/2 pt). Alina's interval $\Delta t'_2$ should also run slow in Fang's frame for the same reason, i.e. $\Delta t'_2 = \sqrt{1 - |v|^2} \Delta t_2 = \Delta t_2 / \gamma$ (1/2 pt). Because Julia and Alina are travelling at the same speed, $\Delta t_1 = \Delta t_2$ (1/2 pt). Noting that Alina's clock was synchronized with Julia's clock after time interval $\Delta t'_1$, we have that $T_{\text{Alina}} = 2\Delta t_1 / \gamma$ (1/2 pt), while $T_{\text{Fang}} = 2\Delta t_1$ (1/2 pt), hence $T_{\text{Alina}} = T_{\text{Fang}} / \gamma$ (1/2 pt). This answer makes sense, since one can think of the 'information' (the time tracked) is NOT in an inertial frame, since it is transferred from Julia to Alina, where both are moving at the same speed but in opposite directions. At the crossing, the 'information' is changing direction, and hence can not be inertial. Alina should register a clock that is running slow compared to Fang's (1 pt).

- (d) (2 pts) Next, let us consider Julia's frame (t', x') . This is a little more involved, and we will try to figure this out in steps. First, since we have not discussed velocity transformations in class, I will simply give you that in Julia's frame, Einstein's velocity transformation will imply that Alina is moving to the left with speed $2|v|/(1 + |v|^2)$. Of course, in Julia's frame, Fang is moving to the left with speed $|v|$. First, let us determine the relation between $\Delta t'_1$ and $\Delta t'_2$. After $\Delta t'_1$, Fang is at a distance $|v|\Delta t'_1$. Alina will catch up with Fang, according to Julia, as

her relative speed with respect to Alina is $2|v|/(1 + |v|^2) - |v|$. From this, show that

$$\Delta t'_2 = \Delta t'_1 \left(\frac{1 + |v|^2}{1 - |v|^2} \right). \quad (4)$$

This is derived from $\Delta t'_2 = \frac{\text{distance}}{\text{velocity}}$ (**1 pts**). The distance is $|v|\Delta t'_1$ (**1/2 pt**) and the velocity is $2|v|/(1 + |v|^2) - |v|$ (**1/2 pt**). Hence we find the relation above.

- (e) (**3 pts**) Once Julia and Alina synchronize their clocks after the first interval, during the second time interval $\Delta t'_2$, Julia then would again argue that Fang's clock is running slow as

$$\Delta t_2 = \sqrt{1 - |v|^2} \Delta t'_2. \quad (5)$$

Show that according to Julia, Alina's clock runs slow as

$$\Delta t''_2 = \frac{1 - |v|^2}{1 + |v|^2} \Delta t'_2, \quad (6)$$

in that same interval.

Recall that in Julia's frame Alina is moving to the left with speed $v'' = 2|v|/(1 + |v|^2)$ (this is a critical part and should receive (**1 pt**)). Hence we have to use this speed instead when we compute the time that transpired on Alina's clock (**1 pt**):

$$\begin{aligned} \sqrt{1 - |v''|^2} &= \sqrt{1 - \frac{4|v|^2}{(1 + |v|^2)^2}} = \sqrt{\frac{(1 + |v|^2)^2}{(1 + |v|^2)^2} - \frac{4|v|^2}{(1 + |v|^2)^2}} \\ &= \sqrt{\frac{1 - 2|v|^2 + |v|^4}{(1 + |v|^2)^2}} = \sqrt{\frac{(1 - |v|^2)^2}{(1 + |v|^2)^2}} = \frac{1 - |v|^2}{1 + |v|^2}, \end{aligned} \quad (7)$$

where we completed the squares to obtain the second equality in the second line on the RHS. Hence according to Julia, Alina's clock should run slow by that factor. This is a hard computation, but will only give (**1 pt**) because it should not outway the reasoning.

- (f) (**3 pts**) The total time, according to Julia, that expired on Fang's clock when Alina reaches Earth should thus be:

$$T_{\text{Fang}} = \sqrt{1 - |v|^2} (\Delta t'_1 + \Delta t'_2), \quad (8)$$

and for Alina:

$$T_{\text{Alina}} = \Delta t'_1 + \frac{1 - |v|^2}{1 + |v|^2} \Delta t'_2. \quad (9)$$

Derive the relation between the arrival time on Earth between Fang and Alina, as determined in Julia's frame. Does this match with the answer you have put in question 2c?

We simply insert all the expressions we obtained. This results in **(1 pt)**:

$$T_{\text{Fang}} = \sqrt{1 - |v|^2} \Delta t'_1 \left[1 + \frac{1 + |v|^2}{1 - |v|^2} \right] = \frac{2\Delta t'_1}{\sqrt{1 - |v|^2}}. \quad (10)$$

And **(1 pt)**

$$T_{\text{Alina}} = \Delta t'_1 + \frac{1 - |v|^2}{1 + |v|^2} \frac{1 + |v|^2}{1 - |v|^2} \Delta t'_1 = 2\Delta t'_1. \quad (11)$$

Hence **(1 pt)**

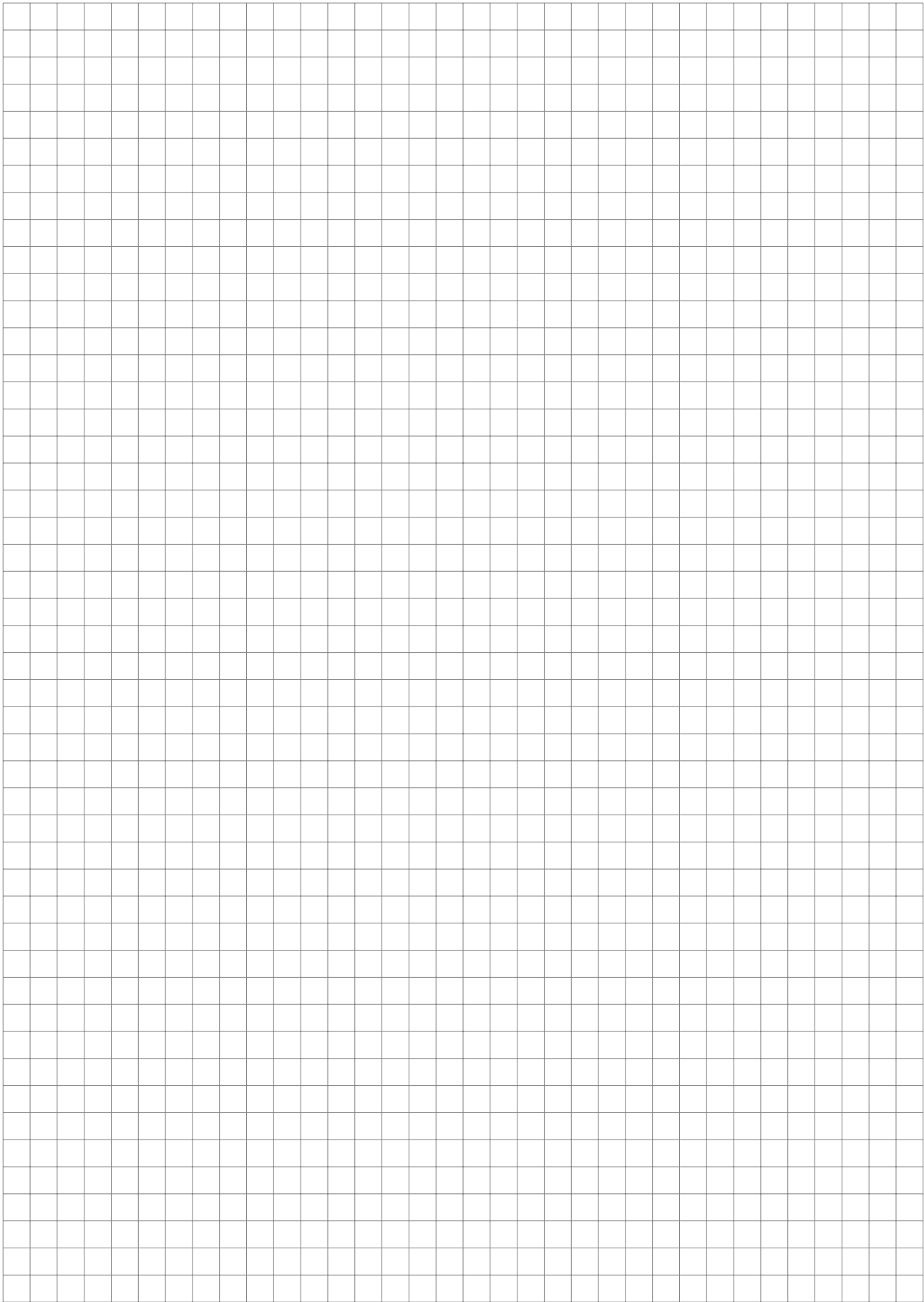
$$T_{\text{Alina}} = \sqrt{1 - |v|^2} T_F = T_{\text{Fang}} / \gamma \quad (12)$$

as before and hence this should have been your answer in 2c. And hence in 2a the student should have said that it is frame independent.

In principle, one can do the same in Alina's frame, but we will leave this for the ambitious student (or for another exam!).

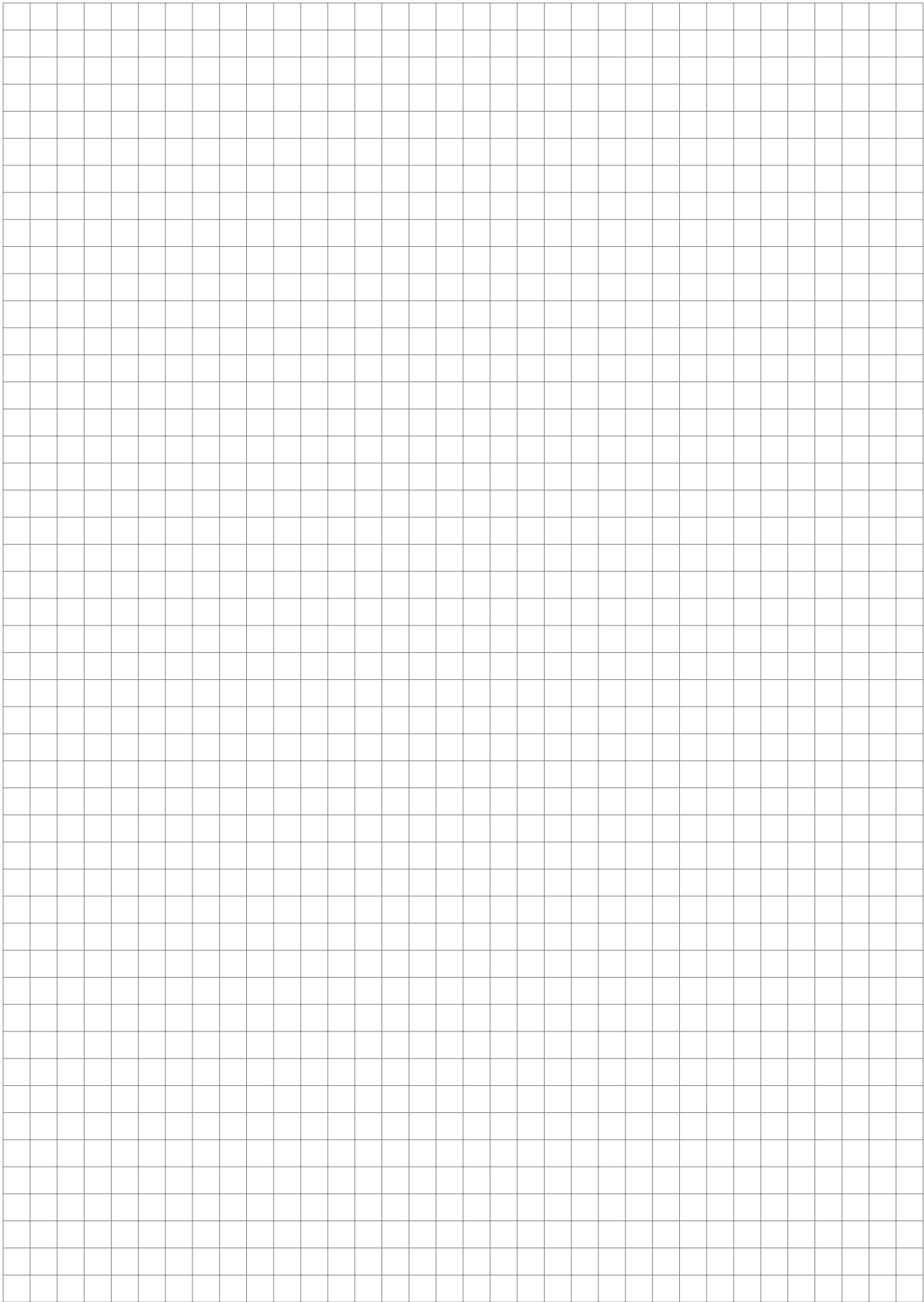
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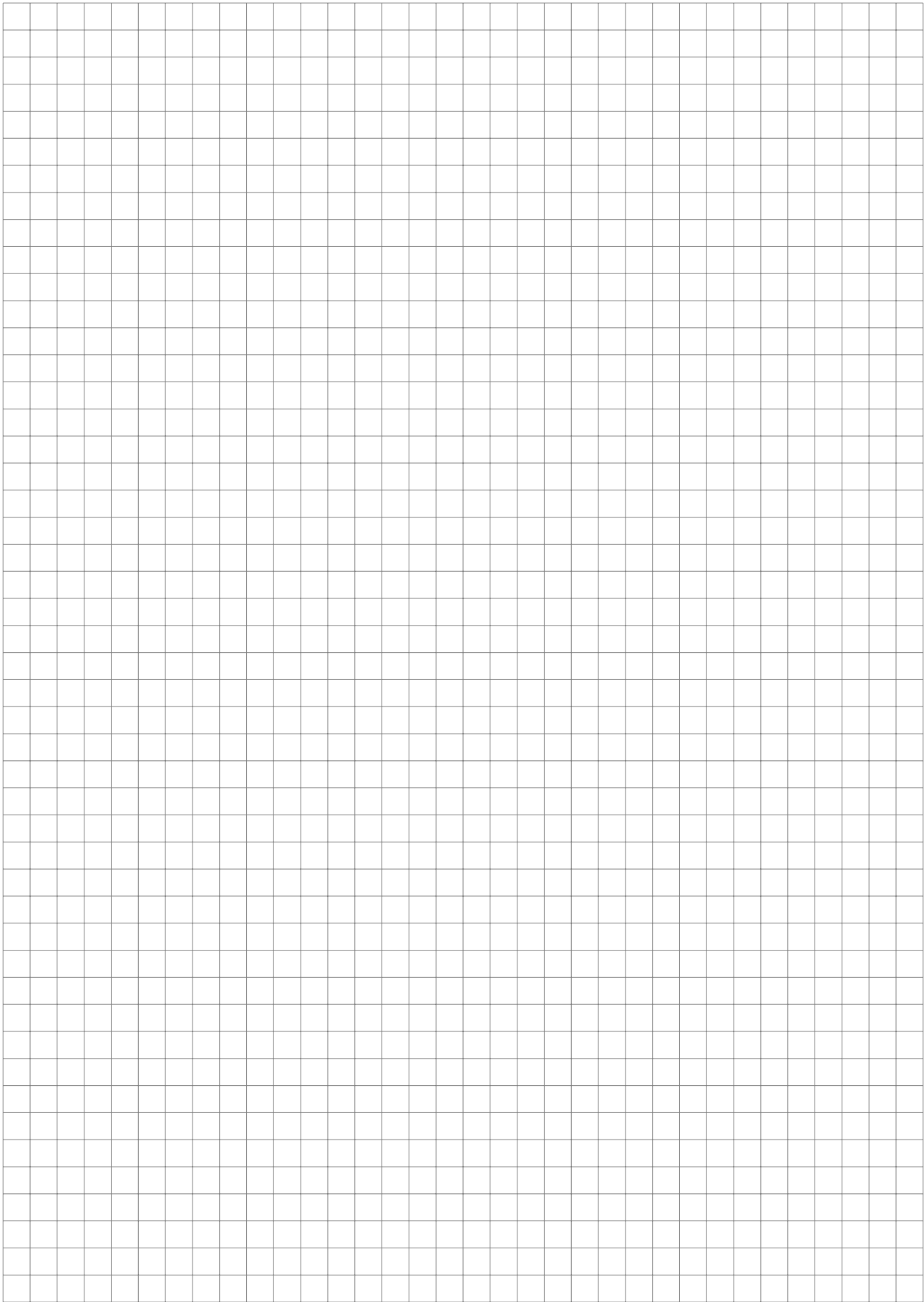
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