Name: Student Number:

Mechanics and Relativity: R2

October 11, 2023, Aletta Jacobshal

Duration: 90 mins

Before you start, read the following:

- There are 2 problems, for a total of 61 points.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

	Points
Problem 1:	36
Problem 2:	25
Total:	61
GRADE (1 + # Total/(9/63))	

Useful equations (SR units):

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$\Delta t \geq \Delta s \geq \Delta \tau$$

The Lorentz transformation equations with $\gamma \equiv (1-\beta^2)^{-1/2}$:

$$\begin{aligned} \Delta t' &= \gamma(\Delta t - \beta\,\Delta x)\,, \\ \Delta x' &= \gamma(\Delta x - \beta \Delta t)\,, \\ \Delta y' &= \Delta y\,, \\ \Delta z' &= \Delta z\,. \end{aligned}$$

Possibly relevant equations:

$$F = G\frac{Mm}{r^2}; \qquad E_k = \frac{1}{2}mv^2; \qquad PV \propto k_b T; \qquad F = \frac{dp}{dt}$$

Possibly relevant numbers (SI Units):

$$c = 299792458 \text{ m/s} \qquad m_H = 1.67 \times 10^{-27} \text{ kg}$$

Question 1: Lorentz Transformations revisited (36 pts)

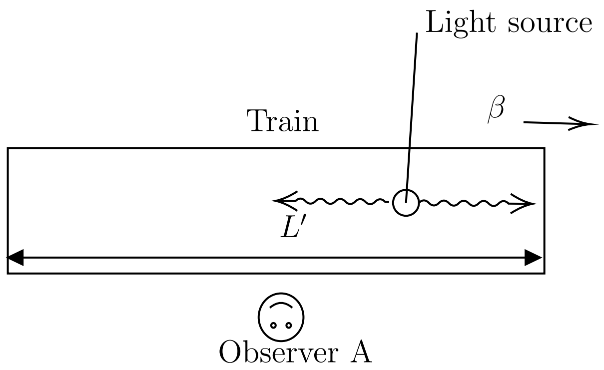
Moore derives the Lorentz transformations using a 2 observer diagram. During the lecture we also derived them by using the invariance of the spacetime interval, as well as time dilation and length contraction. By demanding the transformations to be linear, i.e.

$$\Delta x = A\Delta x' + B\Delta t' \tag{1}$$
$$\Delta t = C\Delta x' + D\Delta t' \tag{2}$$

we were able to solve that $B = C$ and $A = D = \gamma$. From there, we found $B = C = \pm\gamma\beta$, where the $\pm$ can be fixed after deciding how we define the relative motion between the two inertial frames. In this question, we instead will derive relations between A and B and between C and D.

(a) **(3 pts)** First, consider Eq. (1). Suppose I am in an inertial frame (HOME) where $\Delta x = 0$, i.e. I am not moving in that frame. What will be my velocity in a frame moving with β w.r.t. the HOME frame? From there, derive the relation between $\Delta t'$ and $\Delta x'$ and from that an equation for the ratio of B to A (i.e. B/A) in terms of β.

To derive a relation for the ratio of D to C requires a little more work. Instead of not moving, we now consider the limit where two events are simultaneous ($\Delta t = 0$) in the 'HOME' frame. In this limit, let us find a relation between $\Delta x'$ and $\Delta t'$.



For this, let us consider a simple set-up. We will use primed coordinates/quantities for a frame comoving with the train. A train of length L' (at rest) is moving with velocity β w.r.t. to observer **A**. Two light beams are emitted inside the train from somewhere in the train towards the front (right) and the back (left) of the train. Observer **A** looking at the train, observes the two beams to reach the end and front of the train simultaneously ($\Delta t = 0$, unprimed).

(b) **(2 pts)** Argue why the beam appears to reach the front of train with a 'velocity' $c - \beta$ and and the back with $c + \beta$ from the perspective of observer A. Explain why this does not violate the laws of special relativity.

For the two light beams to reach the front and back of the train simultaneously requires that the ratio of distance of the source of the lightbeams of the back (D'_{back}) end to the front (D'_{front}) end has to obey

$$\frac{D'_{\text{back}}}{D'_{\text{front}}} = \frac{c+\beta}{c-\beta} = \frac{1+\beta}{1-\beta}, \tag{3}$$

where we set $c = 1$. At the same time the sum has to equate to the total length of the train, i.e.

$$D'_{\text{back}} + D'_{\text{front}} = L' \tag{4}$$

(c) **(5 pts)** Using these equations, obtain a solution for D'_{back} and D'_{front} in terms of L' and β.

(d) **(2 pts)** Derive the extra distance $D'_{\text{back}} - D'_{\text{front}}$ the beam to the back has to travel in terms of β and L'. How much extra time does this cost?

(e) **(2 pts)** Using this result, argue that if $\Delta t = 0 \to \Delta t' = -\beta \Delta x'$.

(f) **(2 pts)** Derive a relation for C/D in terms of β.

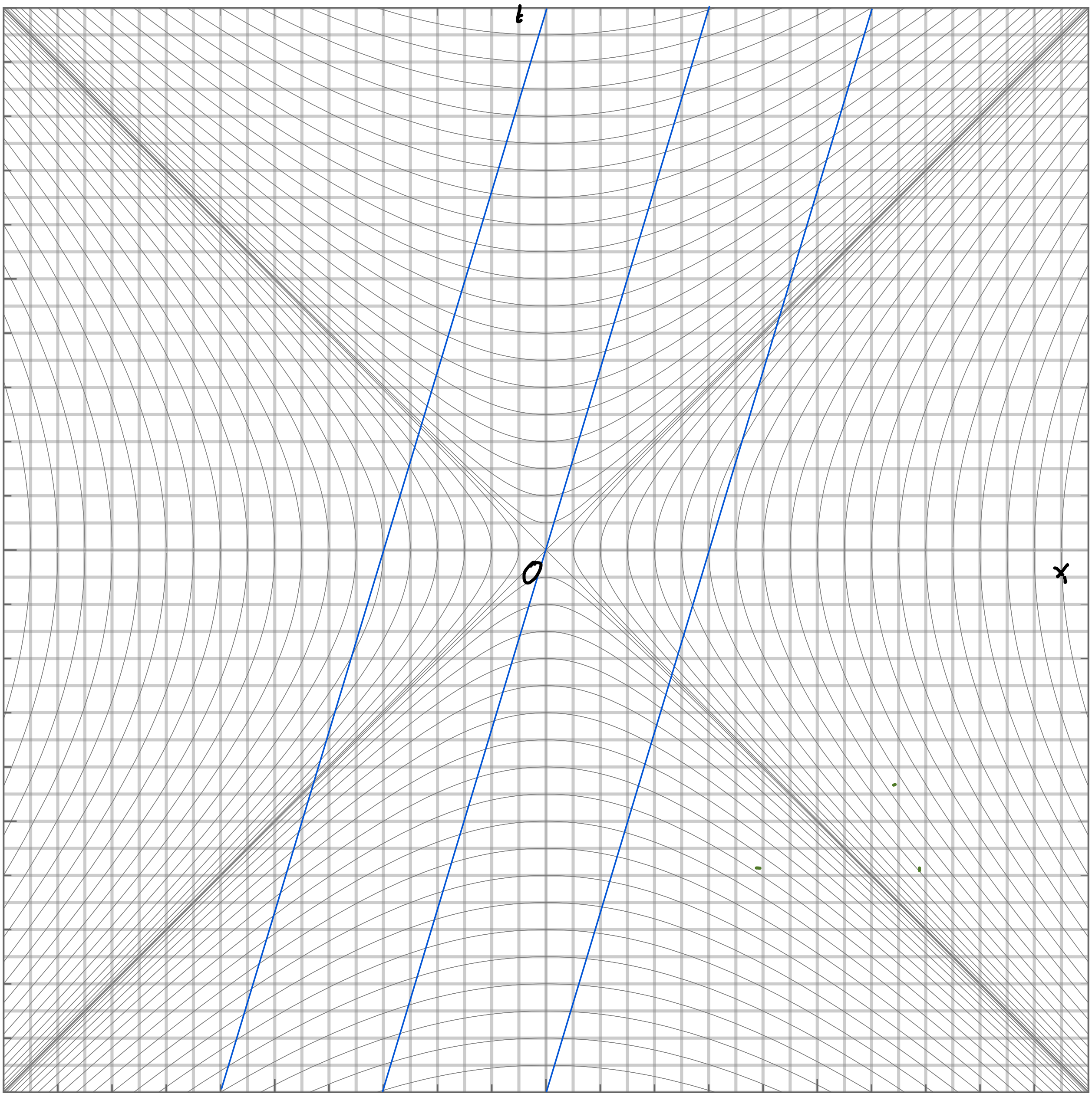
To confirm our results, we can also draw a 2 observer diagram of the train passing by observer A at rest. Assume the center of the train passes by at $t = 0$. For start, we have drawn the worldline of the front, center and back of the train (the train has some arbitrary length L and velocity β).

(g) **(7 pts)** Add the following to the diagram:

- Label the worldline of the front, center and back of the train.
- Label the moment when the front and back of the train are **observed** simultaneously at $t = 0$
- Draw lightrays reaching the front and the back of the train (at $t = 0$). Their intersection with the t-axis (unprimed!) in the past will be 'location' of the source on the train as observed by A.
- Label the length L of the train (in the HOME frame) and the length L' in the train frame

- Draw the time t'-axis and spatial x'-axis of the frame moving with the train
- Label t'_{front} and t'_{back}, the times at which the beams of light reach the front and the back respectively.

Hint: use part of the radar method. The time of emitting the signal will determine the location of the train where the light beams should have been emitted.



(h) **(4 pts)** Use this diagram to read off that indeed the extra time as measured in the moving frame on the clock located at the end of the train will be ahead by $L'\beta$ which implies $\Delta t' = -\beta \Delta x'$. *Hint: Note that the ratio $D_{\text{back}}/D_{\text{front}} = D'_{\text{back}}/D'_{\text{front}}$ because length contraction would apply to both.*

(i) **(9 pts)** Using $\Delta s^2 = \Delta s'^2$ derive the functions A, B, C and D and the final Lorentz transformations. *Hint: In case you could not find an expression for C/D use that $C/D = B/A$.*

Question 2: Tycho's Super Nova (25 pts)

SN 1572 (Tycho's SuperNova) was a supernova of Type Ia in the constellation Cassiopeia, and one of eight supernovae visible to the naked eye in historical records. **It was first seen in 1572.** Currently, the size of the remnant (no longer visible to the naked eye) is ~ 8 arcmin across (60 arc min = 1 degree on the sky).

(a) **(10 pts)** First, using dimensional analysis obtain an expression for the current radius r of the shockwave (the radius of the SuperNova remnant). Assume the relevant parameters are the density ρ_{ISM} of the interstellar medium, the time t the shockfront reaches radius r, and the total energy release of the SuperNova E_{SN}. Use fundamental units L, T and M. *Hint: this problem is similar to the Trinity bomb explosion we derived in class.*

(b) **(3 pts)** Next, show that $D \sim 2r/\Theta$ is the distance to the SuperNova, with Θ the observed angular size of the remnant on the sky and r the current radius of the remnant. *Hint: Use trigonometry by drawing a triangle formed by the SuperNova remnant and Earth. Then use that for small angles Θ, $\tan\Theta \simeq \Theta$.*

(c) **(4 pts)** The density of the interstellar medium is around 1 hydrogen atom per cm^3. The energy release of the SuperNova was about 10^{44} Joules. First, compute the radius of the shockwave today. Assume the dimensionless constant to be 1. Express your answer in lightyears.

(d) **(3 pts)** Now compute the distance to Tycho's SN. Again express your answer in lightyears. *Hint: the above derived relation is true for angles measured in radians!*

NOTE: the next question can be done without the results of the previous questions! More advanced estimates put the distance to the SN remnant at around 6500 lightyears (which could suggest the SN was either less energetic or the ISM is slightly more dense). In any case, an Alien civilization from CasP, a planet in the original stellar system evolving around the dying star, were aware a SuperNova was eminent. They decided to flee to saver worlds and found Earth an appealing option. They left their planet 1000 years (as measured on Earth) before the SuperNova explosion in the direction of Earth at a speed $v = 0.6$ in SR units. *Note that Earth and the star are in the same inertial frame.*

(e) **(5 pts)** How many years after they leave their planet do the Aliens **observe** the SuperNova to go off in their rest frame, assuming the clocks were synchornized at the moment they leave the planet (and thus both the star and the aliens are in inertial frames)? When do they **see** it go off? You can use the 2 observer diagram on the back to sketch the different worldlines.

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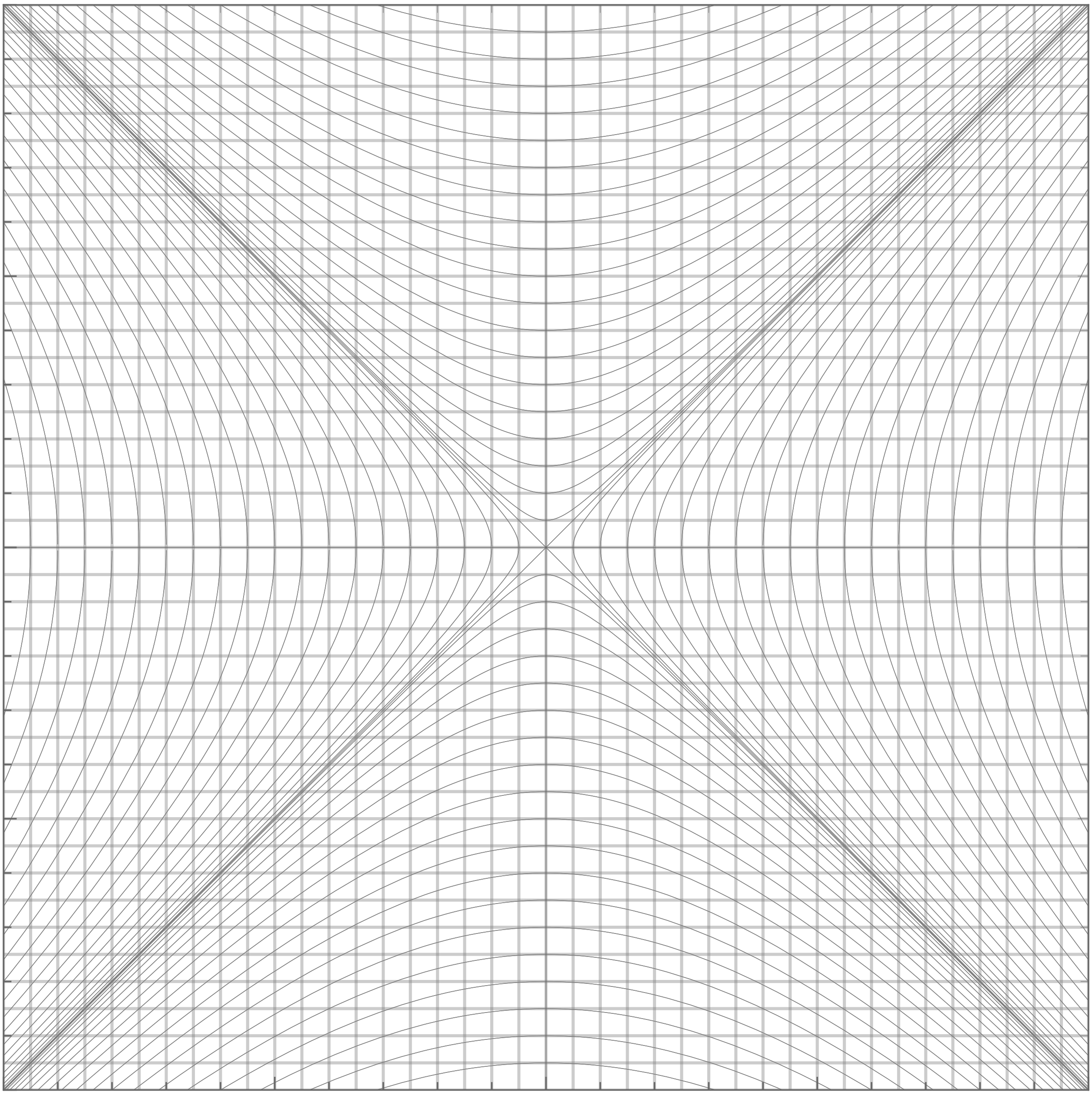

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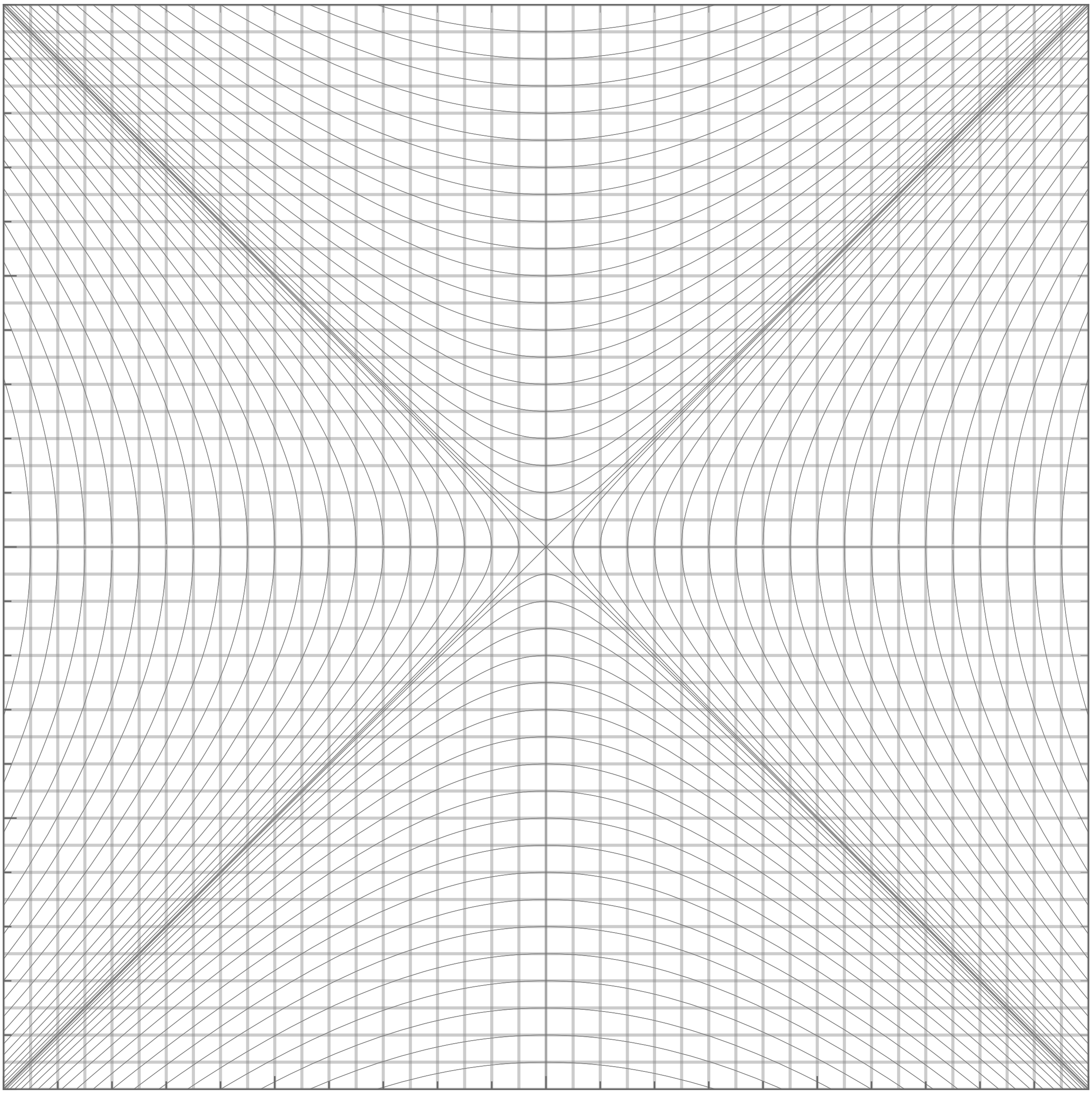

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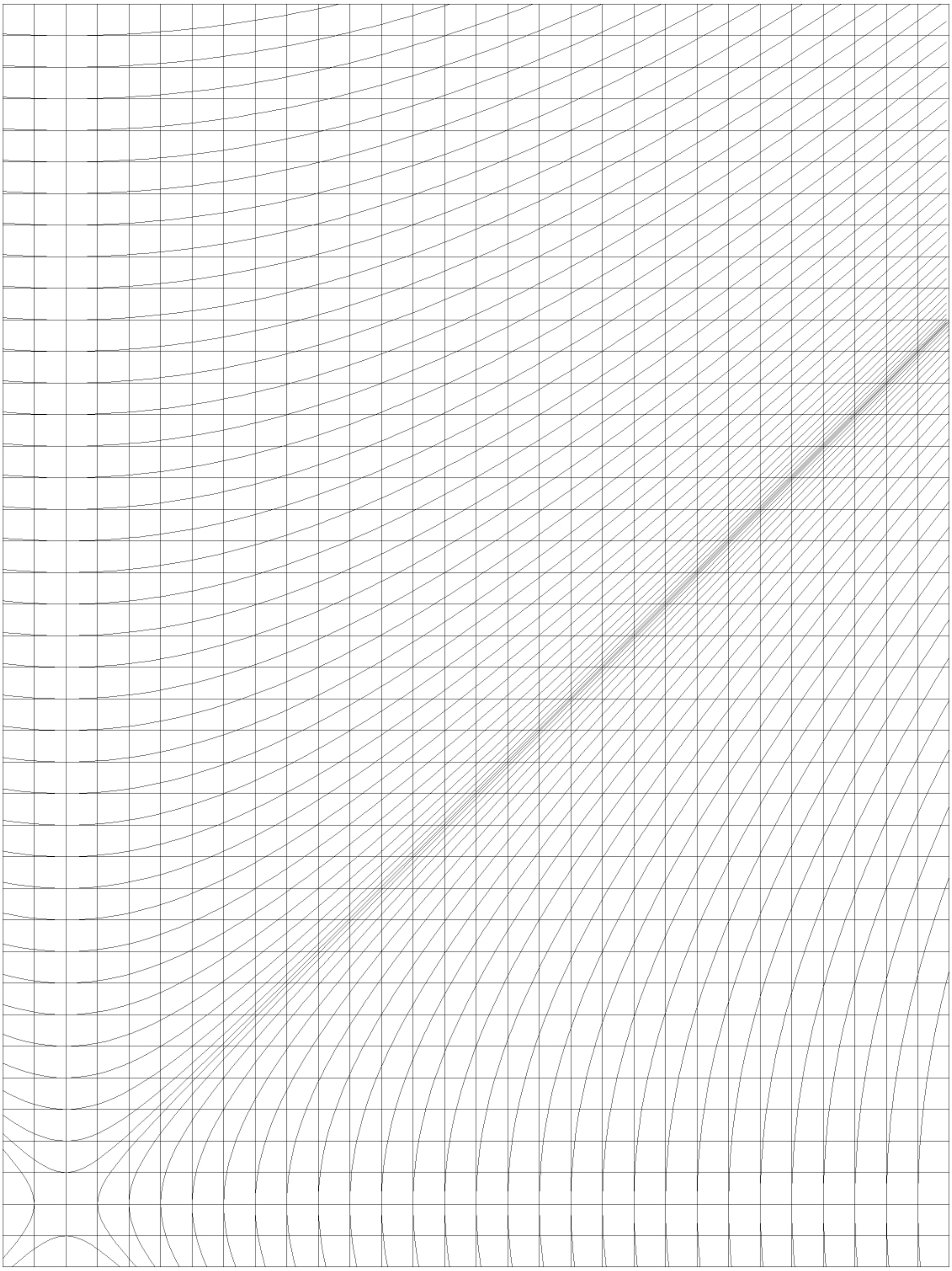

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