Name: Student Number:

Mechanics and Relativity: R2

October 10, 2024, Aletta Jacobshal Exam Hall 2 and 3

Duration: 90 mins

Before you start, read the following:

- There are 3 problems, for a total of 26 points. Question 1d is optional and can be taken for an additional 3 points (which will count as a bonus).
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments with clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you run out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

	Points
Problem 1:	11
Problem 2:	11
Problem 3:	4
Total:	26
GRADE (1 + # Total/(9/26))	

Useful equations (SR units):

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$\Delta t \geq \Delta s \geq \Delta \tau$$

The Lorentz transformation equations with $\gamma \equiv (1-\beta^2)^{-1/2}$:

$$\Delta t' = \gamma(\Delta t - \beta\,\Delta x)\,,$$
$$\Delta x' = \gamma(\Delta x - \beta \Delta t)\,,$$
$$\Delta y' = \Delta y\,,$$
$$\Delta z' = \Delta z\,.$$

The relativistic Doppler shift formula and Einstein velocity transformations:

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1+v_x}{1-v_x}} \quad v'_x = \frac{v_x - \beta}{1-\beta v_x} \quad v'_{y,z} = \gamma^{-1}\frac{v_{y,z}}{1-\beta v_x}\,.$$

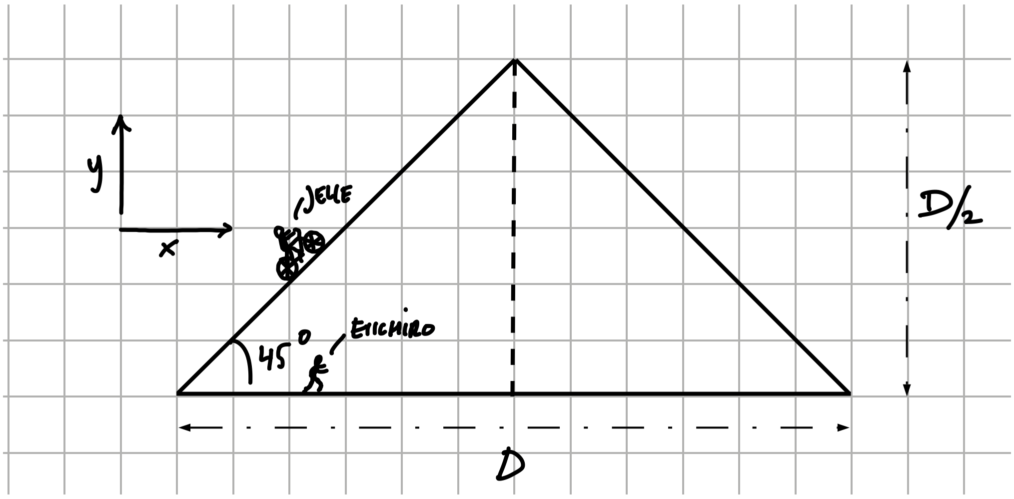

Figure 1: Drawing the correct diagram **(1 pt)**, labelling the distances **(1 pt)**

Question 1: Friends or Foes (11 + 3 pts)
Consider two friends. Jelle is an excellent off-road biker. His friend, Eiichiro, is an outstanding runner. They decide to race each other. To make it fair, Jelle, on his bike, will have to go over a mountain, while Eiichiro will run through a tunnel that goes through the mountain. Assume the length of the tunnel is D, the height of the mountain is $D/2$ and the slope of the mountain is exactly 45 degrees.

(a) **(2 pts)** Sketch the scenario above where tunnel is in x direction and the height of the mountain in the y direction. **Use the checkered paper on the back.** [To be clear, you do not have to draw the time axis.]

(b) **(1 pt)** Assume that both arrive at the same time as measured in the frame at rest compared to Eiichiro and Jelle (HOME frame). Jelle and Eiichiro both bring their own clock, which has impressive sensitivity. Who would argue that he actually won the race? [Just an explanation suffices, no equations]

Jelle, his clock is not inertial and it measures proper time **(1/2 pt)**, which only equates to spacetime in the limit that the clock is inertial. Hence $\Delta\tau < \Delta s$, and Jelle's clock should have run slower compared to Eiichiro's clock **(1/2 pt)**.

(c) **(8 pts)** Consider a scenario where the velocity of Jelle up the mountain is *the same* as down the mountain in the HOME frame. Use $\vec{v} = (v_x, v_y)$ for the velocity of Jelle in the frame at rest and $\vec{u} = (u, 0)$ for the velocity of Eiichiro. In the HOME frame, they arrive at the same time. Now we move to a frame comoving with Eiichiro. In this frame, find an expression for the time passed on Jelle's clock $\Delta t''$ in terms of the time measured on Eiichiro's clock $\Delta t'$ and u, assuming that they synchronized their clocks at the start of the race. *Argue why your derived answer makes sense.*

First, we need to transform the velocities. Moving to a frame where Eiichiro is at rest implies $u' = 0$. The velocity components of Jelle become **(1 pt)**:

$$v'_x = \frac{v_x - u}{1 - uv_x}, \tag{1}$$

$$v'_y = \frac{v_y\sqrt{1-u^2}}{1 - uv_x}. \tag{2}$$

If Jelle is moving up the mountain with the same velocity as down the mountain $u = v_x$ (otherwise Jelle and Eiichiro would not arrive at the same as measured in the rest frame). So $v'_x = 0$ **(1 pt)**. In addition, the vertical displacement is the same $v_x = v_y = u$. So we find that **(1 pt)**

$$v'_y = \frac{u}{\sqrt{1-u^2}}. \tag{3}$$

The time it takes up and down should be equal, and is given by $D/(2u)$ **(1/2 pt)**. Time is dilated in this new frame as $D\sqrt{1-u^2}/(2u)$ **(1/2 pt)**. Since Jelle does not carry an inertial clock we should use proper time **(1 pt)**:

$$\begin{aligned} \Delta t'' \equiv \Delta\tau_J &= 2 \times \sqrt{1 - v'^2_y}\frac{D\sqrt{1-u^2}}{2u} \\ &= 2 \times \sqrt{1 - \frac{u^2}{1-u^2}}\frac{D\sqrt{1-u^2}}{2u} \\ &= \frac{D}{u}\sqrt{1-2u^2}. \end{aligned} \tag{4}$$

For Eiichiro, in the new frame $u' = 0$, but we do have to consider a time dilation which is given by $D\sqrt{1-u^2}/u$, i.e.**(1/2 pt)**

$$\Delta t' = \frac{D\sqrt{1-u^2}}{u}. \tag{5}$$

Hence we find **(2 pts)**:

$$\Delta t'' = \frac{\sqrt{1-2u^2}}{\sqrt{1-u^2}}\Delta t' \tag{6}$$

This answer makes sense since it tells us that for any u, $\Delta t'' < \Delta t'$ as expected **(1/2 pt). If they student uses the v'_y directly into the equation for time dilation, then they should explain why the speed can be considered constant. If they do not, subtract 2pts.** .

(d) **(3 pts)** Show that this agrees with the relation one obtains when considering the HOME frame.

Because the velocity is the same up and down, we only really have to worry about going in one of the two directions to realize the time it takes to go up or down has to be **(1/2 pt)**

$$\frac{D}{2u}. \tag{7}$$

Hence we find that proper time is given by **(1 pt)**:

$$\Delta\tau_J = 2 \times \sqrt{1 - 2u^2}\frac{D}{2u} = \frac{D}{u}\sqrt{1 - 2u^2}. \tag{8}$$

For Eiichiro we also compute proper time **(1 pt)**:

$$\Delta\tau_E = \sqrt{1 - u^2}\frac{D}{u} \tag{9}$$

and **(1/2 pt)**

$$\Delta t'' = \frac{\sqrt{1 - 2u^2}}{\sqrt{1 - u^2}}\Delta t' \tag{10}$$

the same as before.

Question 2: Twin paradox with Longitudinal Doppler (11 pts)

Consider twins Ahmed and Nora. Ahmed uses a spaceship to move back and forth to some distant planet at distance D with constant velocity v. Ahmed and Nora synchronize their clocks ($t = 0$) when Ahmed leaves, and they compare clocks when Ahmed returns. As expected, Ahmed's clock runs slow as $\Delta\tau = dt' = \gamma^{-1}\Delta t$ (since Ahmed is measuring proper time as his clock was not inertial).

In this question, you will use the longitudinal Doppler effect to reproduce this result without explicitly using time dilation (it is hidden in the derivation of the Doppler effect) or Lorentz contraction.

(a) **(1 pt) Sketch** a space-time diagram, comoving with Nora. Draw the worldlines of Nora and Ahmed. Label the coordinate time when he leaves as t_A, the moment Ahmed turns back to Earth as t_B and the time he arrives as t_C (on the coordinate axis). **Use the checkered paper on the back.** *Sketch here means a drawing that is representative of the problem and is drawn with reasonable (consistent with SR) assumptions.*

(b) **(2 pts)** Next, **in that same diagram**, sketch light rays that Nora emits towards Ahmed's spaceship at equal time intervals in coordinate time. Identify which rays will arrive on Ahmed's ship blueshifted and which will arrive redshifted.

(c) **(4 pts)** Next, show that the number of blue or redshifted rays that arrive on Ahmed's ship is given by:

$$N_{\text{RS-rays}} = \frac{D(1-v)}{vdt}, \tag{11}$$

$$N_{\text{BS-rays}} = \frac{D(1+v)}{vdt}, \tag{12}$$

with dt the time between pulses (in the frame comoving with Nora). *Hint: First determine when the last redshifted ray reaches Ahmed before he returns. When was this ray emitted?*

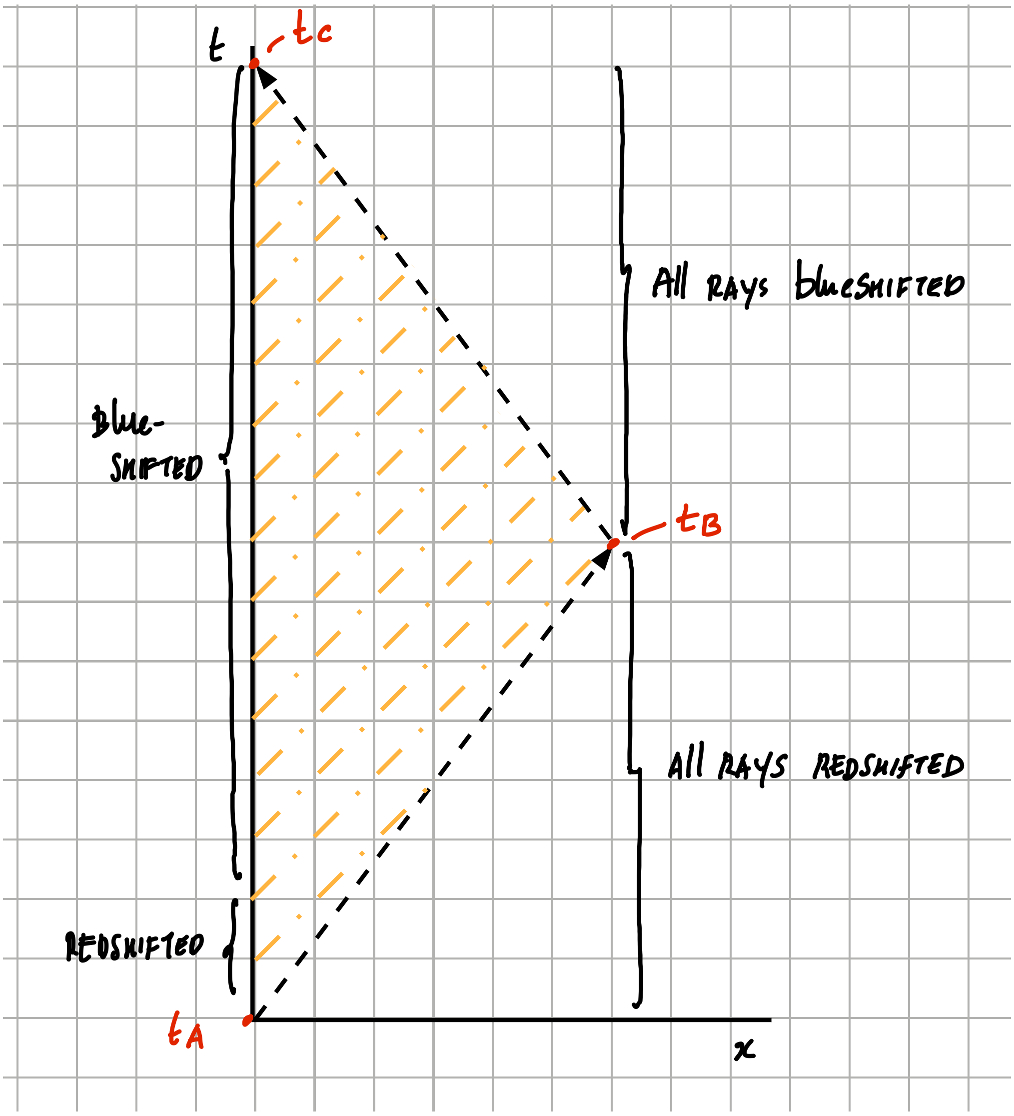

Figure 2: Correct diagram **(1 pt)**. For question b, correct identification of redshifted and blueshifted lines **(1 pt)** and **(1 pt)** for drawing the lines

The last redshifted ray arrives on Ahmed's ship just before he returns. As measured in the frame comoving with Nora this happens at $t_B = D/v$. When did this last ray leave Earth? The worldline of this ray can be described by $t(x) = x + t_{\text{final rs ray}}$. Using that is should be t_B at distance $x = D$ we deduce that **(1 pt)**

$$t_{\text{final rs ray}} = D(\frac{1}{v} - 1) = \frac{D(1-v)}{v}. \tag{13}$$

Assuming the first ray was emitted at $t = 0$, a measure for the total number of rays is thus $\frac{D(1-v)}{v}/dt$, with dt the interval between rays. **(1 pt)**
What about the blueshifted rays? As soon as $t > t_{\text{final rs ray}}$ the rays will arrive blueshifted. The total (coordinate) time Ahmed takes to get back to Earth (this is Nora's clock!) is $t_C = 2t_B = 2D/v$ **(1 pt)**. We should count the number of rays between $t_{\text{final rs ray}}$ and t_C **(1 pt)**:

$$t_C - t_{\text{final rs ray}} = 2\frac{D}{v} - \frac{D(1-v)}{v} = \frac{D(v+1)}{v}, \tag{14}$$

and the number of blueshifted rays is thus $\frac{D(v+1)}{v}/dt$.

(d) **(4 pts)** In class we showed that the time passed between two rays arriving as seen by Ahmed should be related to the time passed in Nora's frame as

$$dt'_{\text{Ahmed}} = \gamma(1+v)dt_{\text{Nora}} \tag{15}$$

(with the opposite sign for blueshifted rays). Use your result from the previous question to derive $\Delta t'_{CA} = \gamma^{-1}\Delta t_{CA}$.

The expression above is for infinitesimal steps; we use that as our steps here. For the redshifted rays, the total time passed on Ahmed's ship is thus **(1 pt)**:

$$\Delta t'_{\text{RS}} = \gamma(1+v)\frac{D(1-v)}{v}\frac{dt}{dt} = \gamma\frac{D(1-v^2)}{v} = \gamma^{-1}\frac{D}{v} \tag{16}$$

and for the blueshifted rays **(1 pt)**:

$$\Delta t'_{\text{BS}} = \gamma(1-v)\frac{D(1+v)}{v}\frac{dt}{dt} = \gamma\frac{D(1-v^2)}{v} = \gamma^{-1}\frac{D}{v} \tag{17}$$

i.e. the same. The total time passed on Ahmed's ship is the sum **(1 pt)**:

$$\Delta t'_{CA} = \Delta t'_{\text{RS}} + \Delta t'_{\text{BS}} = \frac{2D}{v}\gamma^{-1} = \gamma^{-1}\Delta t_{CA}, \tag{18}$$

where we used that $t_C - t_A = \Delta t_{CA} = 2D/v$ **(1 pt)**.

Question 3: Dimensional analysis (4 pts)

(4 pts) In quantum mechanics you will learn that there exists a fundamental relation between the momentum and the displacement of a system which tells us that the product $\Delta x \Delta p \geq \frac{h}{4\pi}$ (here $\geq$ means greater or equal). Use dimensional analysis *as explained in the lecture* (with fundamental units L, M and T) to derive an expression for the energy of the ground state (= lowest possible energy) of such a system of mass m trapped in a well of width D in terms of h, m and D.

Our goal is to find an expression for the energy E. First, we would realize the dimensions of x is $[L]$ and the dimension of p is MLS^{-1}. Hence the dimension of h equals ML^2S^{-1} **(1/2 pt)**. Next the dimension of Energy is $[ML^2S^{-2}]$ **(1/2 pt)**. We then write **(1/2 pts)**:

$$E \propto h^\alpha m^\beta D^\gamma \tag{19}$$

and thus **(1/2 pts)**

$$[ML^2S^{-2}] \propto [ML^2S^{-1}]^\alpha [M]^\beta [L]^\gamma \tag{20}$$

which gives us **(1 pt)**

$$\begin{aligned} S &: -2 = -\alpha \rightarrow \alpha = 2 \\ M &: 1 = \alpha + \beta \rightarrow \beta = -1 \\ L &: 2 = 2\alpha + \gamma \rightarrow \gamma = -2 \end{aligned} \tag{21}$$

which yield the relation **(1 pt)**

$$E = C\frac{h^2}{D^2 m} \tag{22}$$

Noting really that $E = p^2/(2M)$ would then tell you that $C = \frac{1}{4}\frac{1}{(2\pi)^2}$.

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