Name: Student Number:

Mechanics and Relativity: R3

October 30, 2023, Aletta Jacobshal

Duration: 120 mins

Before you start, read the following:

- There are 3 problems, for a total of 55 points and 11 bonus points.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.
- Here $\mathbf{x}$ is a 4-vector, while $\vec{x}$ is a 3-vector. When writing, please make clear when the object is a 4-vector or a 3-vector.

	Points
Problem 1:	23
Problem 2:	30
Problem 3:	13
Total:	66
GRADE (1 + # Total/(9/55))	

Useful equations (SR units):

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$\Delta t \geq \Delta s \geq \Delta \tau$$

The Lorentz transformation equations with $\gamma \equiv (1-\beta^2)^{-1/2}$:

$$\Delta t' = \gamma(\Delta t - \beta\,\Delta x)\,, \qquad \Delta y' = \Delta y\,,$$
$$\Delta x' = \gamma(\Delta x - \beta \Delta t)\,, \qquad \Delta z' = \Delta z\,.$$

The relativistic Doppler shift formula and Einstein velocity transformations:

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1+v_x}{1-v_x}} \quad v'_x = \frac{v_x - \beta}{1-\beta v_x} \quad v'_{y,z} = \gamma^{-1}\frac{v_{y,z}}{1-\beta v_x}\,.$$

Possibly relevant equations:

$$F = G\frac{Mm}{r^2}; \qquad E_k = \frac{1}{2}mv^2; \qquad PV \propto k_b T; \qquad F = \frac{dp}{dt}$$

Possibly relevant numbers (SI Units):

$$c = 299792458 \text{ m/s} \qquad N_{\text{stars}}^{\text{galaxy}} \simeq 2\times 10^{11} \qquad D_{\text{galaxy}} \simeq 10^5 \text{ ly}$$

Question 1: Transformation Aerobics (18 pts + 5 bonus pts)

For the following problems you can consider Δt and Δx only and forget about Δy and Δz.

(a) **(5 pts)** The Lorentz transformations are given on the cover page. Show that by applying the Lorentz transformations to the inverse Lorentz transformations we get back the unprimed coordinate differences Δx and Δt.

(b) **(5 pts)** Next, let us boost from frame S to S' with velocity β and coordinates t' and x' and apply a second boost to a frame S'' with velocity α w.r.t. frame S', with coordinates t'' and x''. As a first step, show that the transformations can be written as

$$\Delta t'' = \gamma'\gamma\left((1+\alpha\beta)\Delta t - (\alpha+\beta)\Delta x\right) \tag{1}$$

$$\Delta x'' = \gamma'\gamma\left((1+\alpha\beta)\Delta x - (\alpha+\beta)\Delta t\right), \tag{2}$$

where

$$\gamma = (1-\beta^2)^{-1/2}, \tag{3}$$

$$\gamma' = (1-\alpha^2)^{-1/2}. \tag{4}$$

(c) **(2 pts)** Frame S' is moving with velocity β w.r.t. frame S. An observer in S' observes an object comoving with frame S'' at velocity α. Use the velocity transformation to obtain the velocity of that object in frame S. Call this velocity κ.

(d) **(5 pts – bonus)** Use this velocity in the Lorentz factor and show that

$$\sqrt{\frac{1}{(1-\kappa^2)}} = \gamma\gamma'(1+\alpha\beta). \tag{5}$$

Use that $1-\alpha^2-\beta^2+\alpha^2\beta^2 = (1-\alpha^2)(1-\beta^2)$.

(e) **(6 pts)** Given Eq. (5) , show that the coordinates in S'' can be obtained via

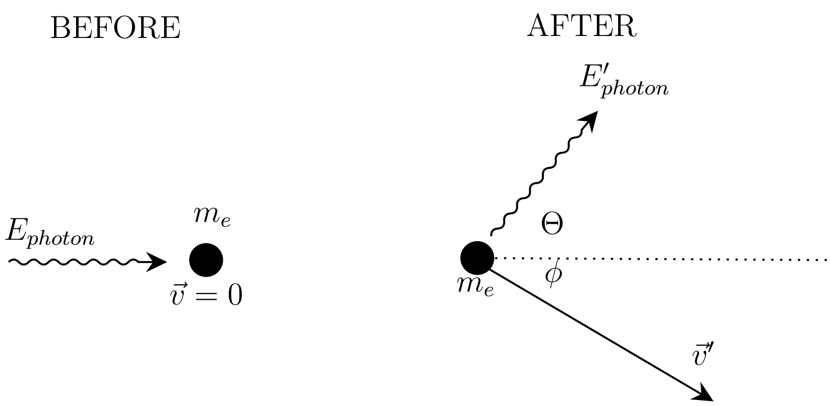
$$\Delta t'' = \gamma''(\Delta t - \kappa \Delta x) \quad (6)$$
$$\Delta x'' = \gamma''(\Delta x - \kappa \Delta t), \quad (7)$$

where

$$\gamma'' = (1 - \kappa^2)^{-1/2}. \quad (8)$$

Explain why this result makes sense.

Question 2: Compton scattering (24 pts + 6 bonus pts)



A photon (particle of light) can scatter of a free electron as shown in the image above, before and after scattering. Assume the electron of mass m_e is at rest. We aim to show that the energy of the photon after scattering is related to the energy of the photon before scattering as

$$E'^{-1}_{\text{photon}} = E^{-1}_{\text{photon}} + m_e^{-1}(1 - \cos\Theta). \tag{9}$$

Note that this answer does not depend on ϕ, i.e. the outgoing angle of the scattered electron.

(a) **(4 pts)** Write down expressions for the 4-momenta of the photon and electron before ($\mathbf{p}_{\text{photon}}, \mathbf{p}_e$) and after ($\mathbf{p}'_{\text{photon}}, \mathbf{p}'_e$) the scattering in terms of E_{photon}, E'_{photon}, γ, γ', Θ, ϕ, m_e and $|\vec{p}|'_e$. Here $|\vec{p}|'_{\text{e}}$ is the magnitude of the relativistic 3-momentum of the electron after the collision.

(b) **(3 pts)** Next, show that this leads to 3 conservation equations:

$$\begin{aligned} \gamma' m &= (E_{\text{photon}} + m - E'_{\text{photon}}), && (10) \\ |\vec{p}|'_e \cos\phi &= (E_{\text{photon}} - E'_{\text{photon}} \cos\Theta), && (11) \\ |\vec{p}|'_e \sin\phi &= E'_{\text{photon}} \sin\Theta. && (12) \end{aligned}$$

(c) **(2 pts)** Now show that

$$\gamma'^2 m^2 - (|\vec{p}|'_e \cos\phi)^2 - (|\vec{p}|'_e \sin\phi)^2 = m^2. \tag{13}$$

(d) **(6 pts)** Combine Eq. (13) with Eqs. (10)-(12) to solve for E'_{photon} (and recover Eq. (9)).

The above solution can be obtained much easier if we simply use that the magnitude of the 4-momenta does not depend on the distribution of relativistic energy and relativistic momentum.

(e) **(2 pts)** What do we know about the magnitude of the electron 4-momentum before ($|\mathbf{p}|$) and after ($|\mathbf{p}'_e|$) the scatter? What about the magnitude of the 4-momentum of the photon?

(f) **(3 pts)** Show that conservation of 4-momentum allows us to derive the following equality:

$$\mathbf{p}_e'^2 = \mathbf{p}_{\text{photon}}^2 + \mathbf{p}_e^2 + \mathbf{p}_{\text{photon}}'^{\,2} + 2\mathbf{p}_e \cdot (\mathbf{p}_{\text{photon}} - \mathbf{p}_{\text{photon}}') - 2\mathbf{p}_{\text{photon}} \cdot \mathbf{p}_{\text{photon}}' \tag{14}$$

(g) **(4 pts)** Use the inner product in Minkowski:

$$A \cdot B = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3. \tag{15}$$

and Eq. (14) to re-derive Eq. (9). Use $E_{\text{photon}} = h\nu = h/\lambda$ to rewrite Eq. (9)

$$\lambda' = \lambda + \frac{h}{m_e}(1 - \cos\Theta), \tag{16}$$

where h is Planck's constant and λ' is the wavelength of the photon after scattering.

(h) **(6 pts – bonus)** Again using conservation of 4-momentum show that

$$\lambda' = \lambda \frac{(1 - v\cos\Theta)}{1 - v} + \frac{h}{\gamma m}\frac{(1 - \cos\Theta)}{1 - v} \tag{17}$$

when the electron has some velocity v to the right before the scattering. Explain why at least part of this result could have been expected.

Question 3: Colonizing the Galaxy (13 pts)

Adam and Eve plan to colonize the Galaxy (typical). On their home planet, they leave behind 2 children, who will populate their home planet. They set-off in a space-ship. Once they reach the nearest star/planetary system they bare 6 more children (3 men and 3 women) and build one additional spaceship. 1 man and 1 woman stay behind, while the two space-ships with 1 man and 1 woman each, fly of in different directions towards the next planetary system. Once they arrive, each of the couple's produce another 6 children and build another space-ship and so on. This repeats, until there are relatives of Adam and Eve on at least one planet around every star in the Galaxy.

Guesstimate how long it will take, as measured by a clock at rest on the home planet, to populate the galaxy with at least 2 humans per planet per star. *Use information provided on the cover page.* To streamline the guestimation, please provide your assumptions on

- The age of man/woman when the leave their home planet.
- The age when they produce new children.
- The number of stars in a typical galaxy.
- The size of a typical galaxy, including shape. Estimate the average volume around a star.
- Estimate the average distance between stars – from there, derive the average speed the space-ship has to fly.

You are allowed to make convenient assumptions, as long as you write them down. *Hint: use that* $a \times a \times a \times ... \times a = a^N$ *and that* $a^N = x$ *can be solved for* N *by taking the* $\log$ *on both sides.*

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