Name: Student Number:

Mechanics and Relativity: R3

October 28, 2024, Aletta Jacobshal, Exam Hall 1

Duration: 120 mins

Before you start, read the following:

- There are 2 problems, for a total of 38 points.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments with clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you run out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.
- Here $\mathbf{x}$ is a 4-vector, while $\vec{x}$ is a 3-vector. When writing, please make clear when the object is a 4-vector or a 3-vector.

	Points
Problem 1:	21
Problem 2:	17
Total:	38
GRADE (1 + # Total/(38/9))	

Useful equations (SR units):

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$\Delta t \geq \Delta s \geq \Delta \tau$$

The Lorentz transformation equations with $\gamma \equiv (1-\beta^2)^{-1/2}$:

$$\Delta t' = \gamma(\Delta t - \beta\,\Delta x)\,, \qquad \Delta y' = \Delta y\,,$$
$$\Delta x' = \gamma(\Delta x - \beta \Delta t)\,, \qquad \Delta z' = \Delta z\,.$$

The relativistic Doppler shift formula and Einstein velocity transformations:

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1+v_x}{1-v_x}} \quad v'_x = \frac{v_x - \beta}{1-\beta v_x} \quad v'_{y,z} = \gamma^{-1}\frac{v_{y,z}}{1-\beta v_x}\,.$$

The binomial formula:

$$(1+x)^n = 1 + nx + \mathcal{O}(x^2)$$

Possibly relevant numbers (SI Units):

$$c = 299792458 \text{ m/s} \qquad N_{\text{stars}}^{\text{galaxy}} \simeq 2\times 10^{11} \qquad D_{\text{galaxy}} \simeq 10^5 \text{ ly}$$

Question 1: A game of space-tennis (21 pts)

A spaceship of proper length $L'_s = 2D$ is leaving Earth in the $+x$ direction with speed $v_\mathrm{s} = \frac{1}{3}$ (in SR units) as measured in the HOME frame. On the ship, two of its passengers are playing a game of high-speed tennis. Assume the tennis ball moves with constant velocity $v'_\mathrm{ball} = \frac{1}{2}$ from the back to the front of the spaceship and back.

(a) **(8 pts)** Draw a space-time diagram including the following:

- The spacetime coordinates of the ground frame
- The spacetime coordinates of the frame comoving with the spaceship
- The worldline of the ball as it hit from the back to the front of the spaceship and returned to the back
- The worldregion of the spaceship

Label the axis in units of D. Assume the ground frame is the HOME frame (as defined in Moore, unprimed coordinates) and the frame comoving with the plane as the OTHER frame (primed coordinates). Assume the back of the spaceship is located at the origin at time $t = t' = 0$. Label the spacetime coordinate event when the ball leaves the back of the spaceship (at time $t = t' = 0$) as **event** A, the spacetime coordinate event when it reaches the front of the spaceship as **event** B and **event** C when it returns to the back. Use the two-observer diagram paper on the right side. If you make a mistake there are additional copies of hyperbolic paper at the back of the exam. Use a pencil to sketch the diagram before you make it permanent.

Hint: Note that to correctly draw the worldline of the ball you need to make sure it has the correct slope in the frame of the spaceship, in which the ball moves with velocity $v'_\mathrm{ball} = \frac{1}{2}$. *As the ball bounces back from the front of the ship, you should realize that the time it takes to reach the back of the spaceship should be the same as reaching the front of the spaceship in the frame comoving with the spaceship.*

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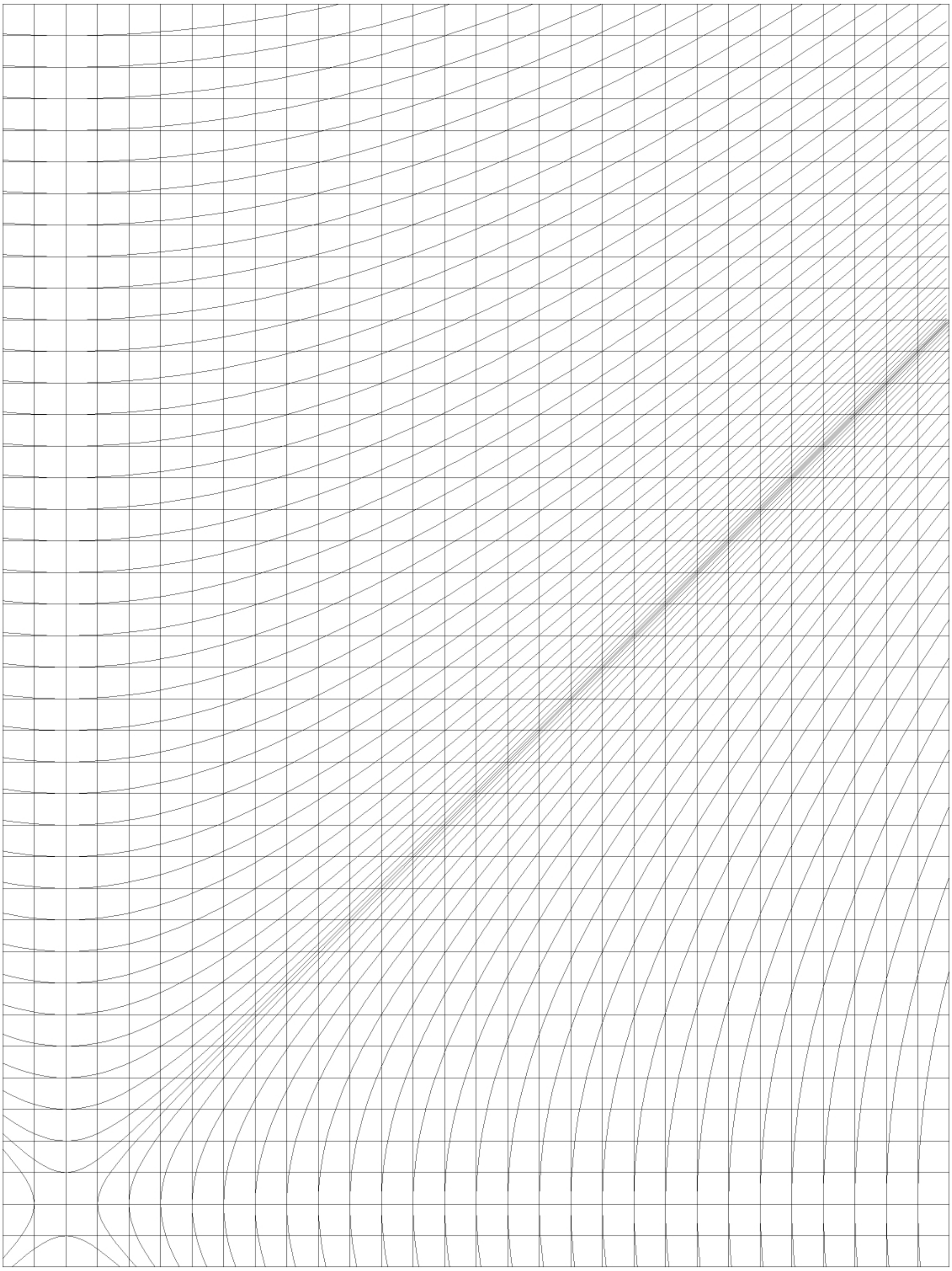

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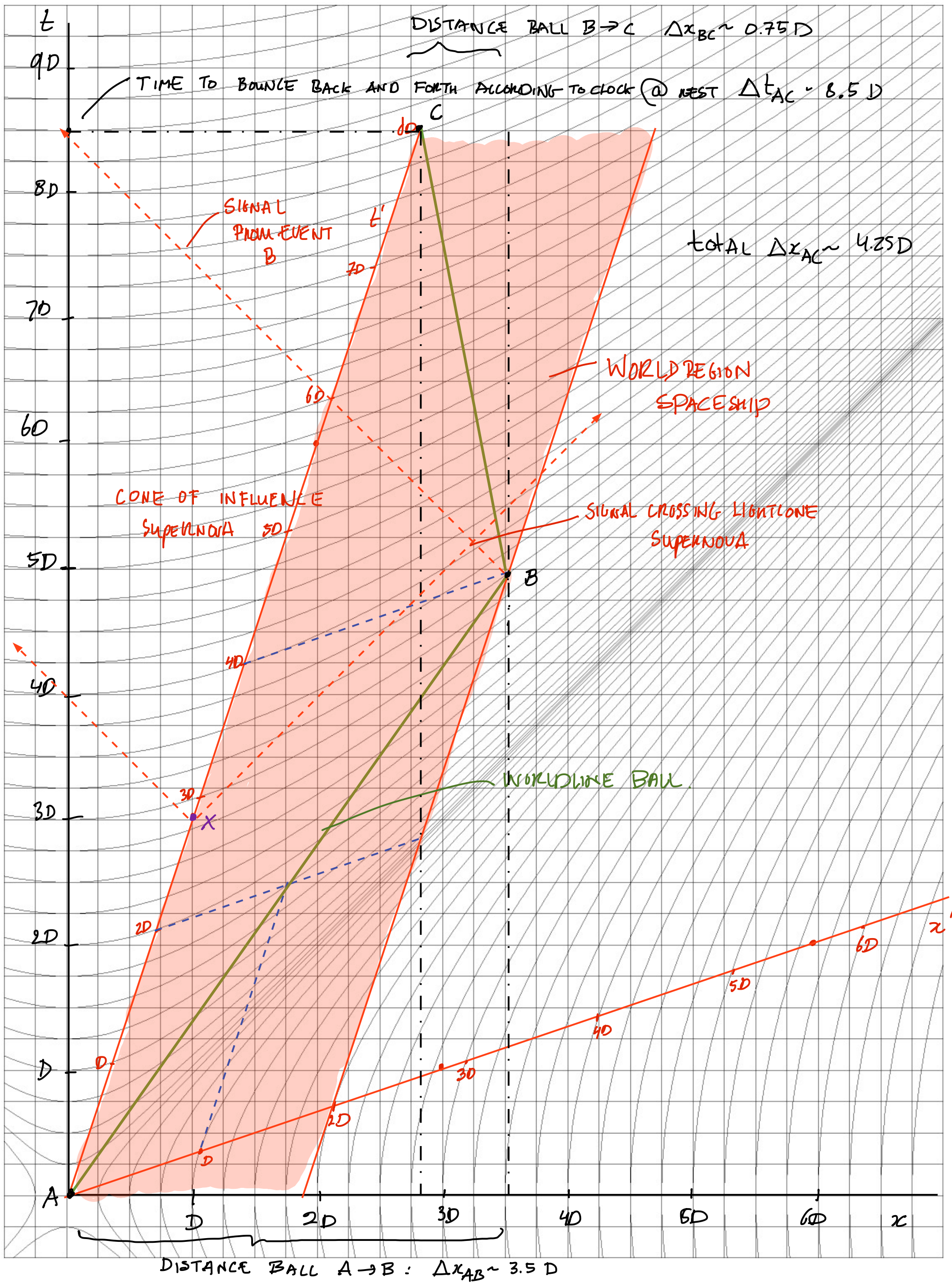
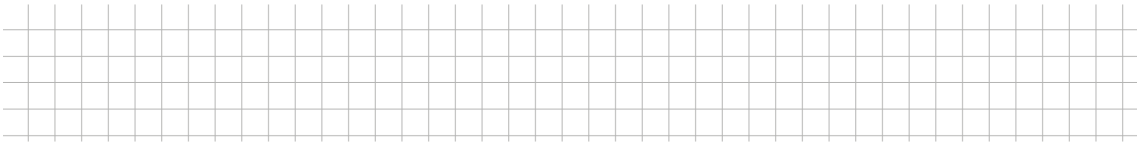




(1 pt): ground frame axis
(1 pt): spaceship frame axis
(1 pt): worldline ball
(1 pt): world region spaceship
(1 pt): ticks using D
(1 pt each): labeling events A, B and C

(b) **(2 pts)** Based on your space-time diagram, as observed from the ground (HOME) frame, read off

- the time it takes for the ball to move back and forth through the spaceship in units of D, i.e. the time elapsed between events A and C as measured on a clock at rest
- the distance the ball covers (the absolute sum of back and forth) in that time in units of D, i.e. the distance between events A and B summed with the (absolute) distance between B and C as measured on a ruler at rest

If your space-time diagram is correct, you should be able to obtain the answer with $\sim 10\%$ accuracy.

It is straightforward to read of the $\Delta t_{AC} \sim 8.5\text{D}$ **(0.5 pt)**. For the ball's distance, we first read the distance $\Delta x_{AB} \sim 3.5\text{D}$ **(0.75 pt)**. Then we read off the distance between $\Delta x_{BC} \sim 0.75D$. The total is thus $\Delta x_{AC} \sim 4.25\text{D}$ **(0.75 pt)**

(c) **(1 pt)** Answer the same question but now for the frame comoving with the spaceship (OTHER).

The distance the ball covers is obviously just the length of the spaceship times 2, i.e. $4D$ as observed in the frame comoving with the plane **(0.5 pts)**. The time it takes can be read off, but also simply computed as **(0.5 pts)**

$$\Delta t'_{AC} = 2 \times \frac{2D}{v'_{\text{ball}}} = 8D \tag{1}$$

(d) **(6 pts)** To confirm your answer to question (b) (HOME frame), compute the distance and time covered in the HOME frame, by using velocity transformations and length contractions. This is a multi-step problem, so use the following hints to your advantage:

- We first need to obtain v_{ball} (the speed of the ball in the HOME frame). However, you need to separate the part where v'_{ball} is forwards $A \to B$ and when it is backwards $B \to C$. If you can't find an answer, use $v_{\text{ball,AB}} = \frac{5}{7}$ and $v_{\text{ball,BC}} = -\frac{1}{5}$.
- In the HOME frame the length of the spaceship is contracted. If you can not find an answer, use $L_{\text{s}} = \frac{4\sqrt{2}}{3}D$. Recall that the length of the spaceship at rest is $L'_s = 2D$.
- As the spaceship is moving in the positive x direction with $v_{\text{s}} = \frac{1}{3}$, you have to consider the relative speed of the ball w.r.t. the HOME frame, as the front of the ship is moving away from the ball and the back of the ship is moving towards the ball ('closing the gap').

We first need to obtain v_{ball} (the speed of the ball in the HOME frame). However, we have to separate the part where v'_{ball} is forwards $A \to B$ and when it is backwards $B \to C$. From the cover page, we have the Einstein velocity transformations. However, we should replace the $-$ with $+$ since we go from OTHER to HOME **(0.5 pts)**. We have $v'_{\text{ball,AB}} = 1/2$ and $\beta = 1/3$

$$v_{\text{ball,AB}} = \frac{v'_{\text{ball,AB}} + \beta}{1 + \beta v'_{\text{ball,AB}}} = \frac{1/2 + 1/3}{1 + 1/6} = \frac{5}{7} \ (\mathbf{0.5\ pts}). \tag{2}$$

Next, for the velocity as the ball is returned to the back of the spaceship:

$$v_{\text{ball,BC}} = \frac{v'_{\text{ball,BC}} + \beta}{1 + \beta v'_{\text{ball,BC}}} = \frac{-1/2 + 1/3}{1 - 1/6} = -\frac{1}{5} \ (\mathbf{0.5\ pts}). \tag{3}$$

Next, we need to obtain the length of the plane as observed from the ground. As observed from the ground, the plane should be contracted via **(0.5 pts)**

$$L_s = 2D/\gamma_{1/3} = 2D\sqrt{1 - \beta^2} = 2D\sqrt{1 - (1/3)^2} = \frac{4\sqrt{2}}{3}D \tag{4}$$

Since the spaceship is moving with speed 1/3, as the ball moves towards the front of the spaceship, the front of the spaceship is also moving forward ($v_{\text{plane}} \equiv \beta = 1/3$) as observed in the ground frame. We should therefore account for this by noting that the relative speed of the ball with respect to the front of the plane is given by $v_{\text{rel,AB}} = v_{\text{ball,AB}} - v_{\text{plane}} = 5/7 - 1/3 = 8/21$ **(0.5 pts)**. The time it takes for the ball to reach the front of the plane as observed from the ground is therefore **(0.5 pts)**:

$$\Delta t_{\text{ground,AB}} = L_s/v_{\text{rel,AB}} = \frac{4\sqrt{2}}{3} * 21/8D = \frac{7}{\sqrt{2}}D \simeq 5D. \tag{5}$$

This is exactly what we found from reading this off in the spacetime diagram. The distance can be obtained by multiplying the time with the speed of the ball **(0.5 pts)**:

$$\Delta x_{\text{ground,AB}} = v_{\text{ball,AB}} * \Delta t_{\text{ground,AB}} = 5/7 * \frac{7}{\sqrt{2}}D = 5D/\sqrt{2} \simeq 3.5D \tag{6}$$

Again this is consistent with what we read off directly from the diagram. Next, for the ball moving back to the spaceship, we do the same steps, $v_{\text{rel,BC}} = |v_{\text{ball,BC}} + v_{\text{plane}}| = 1/5 + 1/3 = 8/15$ **(0.5 pts)** (the back is moving towards the ball!!): **(0.5 pts)**:

$$\Delta t_{\text{ground,BC}} = L_s/v_{\text{rel,BC}} = \frac{4\sqrt{2}}{3} * 15/8D = \frac{5}{\sqrt{2}}D \simeq 3.5D. \tag{7}$$

and the displacement **(0.5 pts)**:

$$\Delta x_{\text{ground,BC}} = v_{\text{ball,BC}} * \Delta t_{\text{ground,BC}} = 1/5 * \frac{5}{\sqrt{2}}D \simeq 0.7D \tag{8}$$

and hence the total distance the ball has travelled between event A and C is $\Delta x_{AC} = \frac{6}{\sqrt{2}}D \sim 4.35D$ **(0.5 pts)**. The total time is $\Delta t_{AC} = \frac{12}{\sqrt{2}}D \simeq 8.5D$**(0.5 pts)**

(e) **(2 pts)** Now confirm the answer to questions (*b*) and (*d*) by using the Lorentz transformations to go from the frame comoving with the plane to the ground frame. *Hint: use your results from question (c) for* $\Delta t'_{AC}$ *and the fact that the ball has zero net displacement as measured in the frame comoving with the spaceship. Note that this is NOT the same as the absolute displacement, which is never zero unless you are in a frame comoving with the ball.*

The Lorentz transformations are given on the cover page, but since we go from the OTHER frame to the HOME frame we should replace $\beta \to -\beta$ **(0.5 pts)**. From question (*c*) we got $\Delta t'_{\rm AC} = 8D$ and $\Delta x'_{\rm AC} = 0$ (since after a bounce back and forth there is no net displacement in that frame). Applying the Lorentz transformations thus gives **(1 pt)**:

$$\Delta t_{\rm ground,AC} = \gamma_{1/3}(\Delta t'_{\rm AC} + \beta\Delta x'_{\rm AC}) = \frac{3}{2\sqrt{2}}8D = 12D/\sqrt{2} \tag{9}$$

For the total displacement of the ball in the home frame we have to be a little careful. We should first compute the displacement going forward with **(1pt, 0.5 pt each)**:

$$\Delta x_{\rm ground,AB} = \gamma_{1/3}(\Delta x'_{\rm AB} + \beta\Delta t'_{\rm AB}) = \frac{3}{2\sqrt{2}}(2D + 4D/3) = 5D/\sqrt{2} \tag{10}$$

$$\Delta x_{\rm ground,BC} = \gamma_{1/3}(\Delta x'_{\rm CB} + \beta\Delta t'_{\rm CB}) = \frac{3}{2\sqrt{2}}(-2D + 4D/3) = -D/\sqrt{2} \tag{11}$$

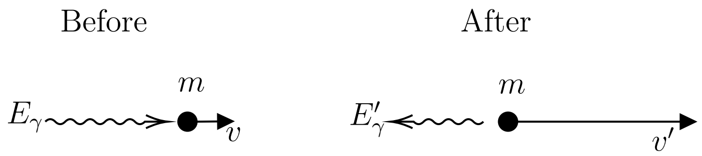
The displacement is negative, but for the total displacement by the ball, it should be added **(0.5 pts)**. The results agree with what we found in (b) and (d).

(f) **(2 pts)** An observer on Earth is following the tennis on her local TV through a live broadcast. Just before the first time the ball comes back from the front of the ship (event B), the transmission is interrupted and her screen goes black. Just a while before, she **observed** a Supernova go off at $x = D$ and $t = 3D$. Label this event X in your diagram. Could event X causally affect event B? *If you did not find a solution for the coordinates of event B, use* $(t_B, x_B) = (5D, 3.5D)$. Is the interval between these events time-like, light-like or space-like? Can the Supernova cause the interruption of the TV signal? *Hint: draw the lightcone of the Supernova and the worldline of the signal towards Earth coming from event B.*

No, if you draw the lightcone of the supernova, you will find it can not have affected event B **(0.5 pts)**. The interval between events X and B is space-like **(0.75 pts)**. However, as the signal propagates to Earth before it reaches Earth, it passes through the lightcone of the supernova. Hence, the signal itself could have been affected **(0.75 pts)**. If students state that this is unlikely since the signal could have been affected even before event B, that is also ok.

Question 2: Are we alone in the Universe? (17 pts)

Consider the following elastic interaction between a photon and a particle of mass m:



(a) **(2 pts)** Write down the 4-vectors of the photon before and after the elastic scattering. Use E_γ (and primed coordinates). Similarly write down the 4-vector of the particle before and after in terms of its mass, its velocity v and v' and the Lorentz factors γ and γ'.

Before the interaction, the 4-vectors are given by **(1 pt, 0.5 pt each)**

$$\mathbf{p}_\gamma = (E_\gamma, E_\gamma, 0, 0) \quad (12)$$
$$\mathbf{p}_m = (\gamma m, \gamma m v, 0, 0), \quad (13)$$

while after they are given by **(1 pt, 0.5 pt each)** :

$$\mathbf{p}'_\gamma = (E'_\gamma, -E'_\gamma, 0, 0) \quad (14)$$
$$\mathbf{p}'_m = (\gamma' m, \gamma' m v', 0, 0), \quad (15)$$

where

$$\gamma' = \frac{1}{\sqrt{1 - v'^2}} \quad (16)$$

(b) **(2 pts)** Show (derive) that conservation of 4-momentum allows us to write:

$$2E_\gamma = m(\gamma' - \gamma) + m(\gamma' v' - \gamma v). \quad (17)$$

Equating the components before and after the interaction leads to **(1 pt, 0.5 pt each)** :

$$E_\gamma + \gamma m = E'_\gamma + \gamma' m \quad (18)$$
$$E_\gamma + \gamma m v = -E'_\gamma + \gamma' m v'. \quad (19)$$

Summing them, and bringing all terms involving the mass and velocity of the spaceship to the right-hand side yields **(1 pt)**:

$$2E_\gamma = m(\gamma' - \gamma) + m(\gamma' v' - \gamma v). \quad (20)$$

(c) **(4 pts)** Now we want to compute the total energy required to reach velocity β after a (large) number of photons has hit the particle. For this, we rewrite the above equation as:

$$2dE = m d\gamma + m d(\gamma v). \quad (21)$$

where d now presents a small change by hitting the particle with a photon $dE = E_\gamma$. Now use that we can write[1]:

$$d\gamma = \frac{d\gamma}{dv} dv, \tag{22}$$

where $d\gamma/dv$ now is the derivative of γ with respect to v, and the chain rule that

$$2dE = m(v\gamma^3 + v^2\gamma^3 + \gamma)dv = m\frac{\gamma}{1-v}dv. \tag{23}$$

We need to obtain the derivative of γ w.r.t. the velocity v (done in class) **(1 pt)**:

$$\frac{\partial \gamma}{\partial v} = \frac{\partial}{\partial v}(1-v^2)^{-1/2} = v(1-v^2)^{-3/2} = v\gamma^3. \tag{24}$$

The chain rule states that (done in class) **(0.5 pts)**

$$d(\gamma v) = v d\gamma + \gamma dv, \tag{25}$$

hence we find, using Eq. (24) **(0.5 pts)** :

$$2dE = m(v\gamma^3 + v^2\gamma^3 + \gamma)dv. \tag{26}$$

Then **(0.75 pts)**

$$\gamma + v^2\gamma^3 = \gamma(\gamma^2 v^2 + 1) = \gamma\left(\frac{v^2}{1-v^2} + 1\right) = \gamma\left(\frac{v^2}{1-v^2} + \frac{1-v^2}{1-v^2}\right) = \gamma^3 \tag{27}$$

and **(0.75 pts)**

$$(v\gamma^3 + \gamma^3) = \gamma\gamma^2(1+v) = \gamma\frac{1+v}{1-v^2} = \gamma\frac{1+v}{(1-v)(1+v)} = \frac{\gamma}{1-v}. \tag{28}$$

Combined this results in **(0.5 pts)**

$$2dE = m\frac{\gamma}{1-v}dv \tag{29}$$

(d) **(2 pts)** Next, integrate this expression to show that

$$E_{\text{tot}} = \frac{m}{2}\left[\sqrt{\frac{1+\beta}{1-\beta}} - 1\right], \tag{30}$$

with β the final speed of the particle. You can use:

$$\int_0^A \frac{1}{\sqrt{(1-x)^3(1+x)}}dx = \sqrt{\frac{1+A}{1-A}} - 1 \tag{31}$$

[1] Technically we should write $\frac{\partial \gamma}{\partial v}dv$, with ∂ representing partial derivatives. This will be introduced in Calculus 2. The expression here is correct as long as there are no other dependencies but v which is the case.

Argue why this result makes sense (3 sentences max).

Realizing that $\gamma/(1-v) = ((1-v)^3(1+v))^{-1/2}$, which integrand above, and setting $A = \beta$ we write **(0.5 pts)** :

$$2\int_0^{E_{\rm tot}} dE = m\int_0^{\beta} \frac{\gamma}{1-v}dv = m\left[\sqrt{\frac{1+\beta}{1-\beta}} - 1\right], \tag{32}$$

and divide by two to obtain **(0.5 pts)** :

$$E_{\rm tot} = \frac{m}{2}\left[\sqrt{\frac{1+\beta}{1-\beta}} - 1\right]. \tag{33}$$

The answer makes sense since it is 0 when $\beta = 0$, it looks like total redshifted energy and this also means I can not exceed $\beta = 1$ unless I inject an infinite amount of energy. The total energy also depends on the mass, i.e. the more massive the spaceship the more energy is required **(1 pt)** .

(e) **(1 pt)** Show that for small velocities ($v \ll c$) this reduces to:

$$E_{\rm tot} = \frac{1}{2}m\beta + \mathcal{O}(\beta^2). \tag{34}$$

We can use the binomial formula (front page):

$$(1+x)^n = 1 + nx + \mathcal{O}(x^2) \tag{35}$$

to find that **(0.5 pts)**

$$\sqrt{\frac{1+x}{1-x}} = (1 + \frac{1}{2}x + \mathcal{O}(x^2))(1 + \frac{1}{2}x + \mathcal{O}(x^2)) = 1 + x + \mathcal{O}(x^2) \tag{36}$$

Using this we find **(0.5 pts)**

$$E_{\rm tot} = \frac{1}{2}m\beta + \mathcal{O}(\beta^2) \tag{37}$$

The equation makes sense since it looks like normal momentum conservation which would be sufficient to estimate the energy required to reach speed β.

(f) **(6 pts)** The Breakthrough Starshot mission to travel to Alpha Centauri – the closest star to our solar system – was announced in 2016 and invited researchers to design a spaceship powered by light. The idea is to have a minute spacecraft (without humans!) that has sails which are propelled not mechanically or by wind, but instead by a laser that is sending light to hit the sails. This laser is stationed on Earth. The spaceship has to be very light (about the weight of a ballpoint pen/pencil). We aim to guesstimate the power of the laser necessary to reach the targeted speed of a spaceship – which itself is determined to be able to reach Alpha Centauri (AC) within our lifetimes.

Using the equation (see (d) or (e)) for the total amount of energy to accelerate an object of mass m to a speed β, guesstimate the power in Watt (Joules per second) of the laser required to reach this speed within 10 minutes. We suggest the following approach:

- Guesstimate the speed required by making an assumption about the distance to AC (in lightyears!) and our expected 'lifetime'.
- Make a reasonable assumption about the mass of the spaceship.

- The nearest star is a couple of lightyears away, assume 5. You can also guesstimate this from the cover page by assuming the stars in the galaxy are distributed either on a surface or in a cube. For the former you take $\sqrt{N_{\text{stars}}} \simeq 10^5$. Hence you would find a star roughly every light year (given the diameter). This is a bit on the high side because you just assumed stars are on a plane. If you assume they are in a cube you would get a star every 20ly's (which is a bit low) – anywhere between 1-20 would be ok. **(1 pt)**
- Our lifetime is about 50 years (left), hence we want this spaceship to reach AC in 50 years. Hence we should reach 1/10 of the speed of light, i.e. $\beta = 0.1$. **(1 pt)**
- In the intro we stated that the spaceship should be a few grams, assuming 10 (but maybe 100 or 5 is also fine). Not all the expressions above are in SR units and we have to multiply the above by c^2 to get back to SI units. The total required energy is thus **(0.5 pts)** :

$$E_{\text{tot}} \simeq 10^{-2}\text{kg} \times (\sqrt{\frac{1.1}{0.9}} - 1)c^2 \tag{38}$$

- Note that we have to be a little careful here, since if we approximate $1.1/0.9 \simeq 1$, the whole thing is zero. Better is to assume the whole thing is order 10^{-1}. Note that this is only true because $\beta = 0.1$. If we had to speed up to $\beta = 0.9$, we get order 1, and 0.99 order 10. Students can also use the limit of $v \ll c$ and that should lead to the same answer (up to a factor of 2) **(0.5 pts)**.
- Assuming order 10^{-1}, and using $c \sim 3 \times 10^8$ m/s, we find $E_{\text{tot}} \sim 10^{14}$ Joules required **(1 pt)**. If we want to reach this speed in 10 min $= \Delta t = 60 \times 10 \sim 5 \times 10^2$ sec **(1 pt)** , we guesstimate the power of the laser to be $E_{\text{tot}}/\Delta t \simeq 2 \times 10^{11}$ Watts, or about 100 GW (giga watts) **(1 pt)** . This is the same as what is envisioned. Note that this can not be achieved with a single laser and requires a laser array.

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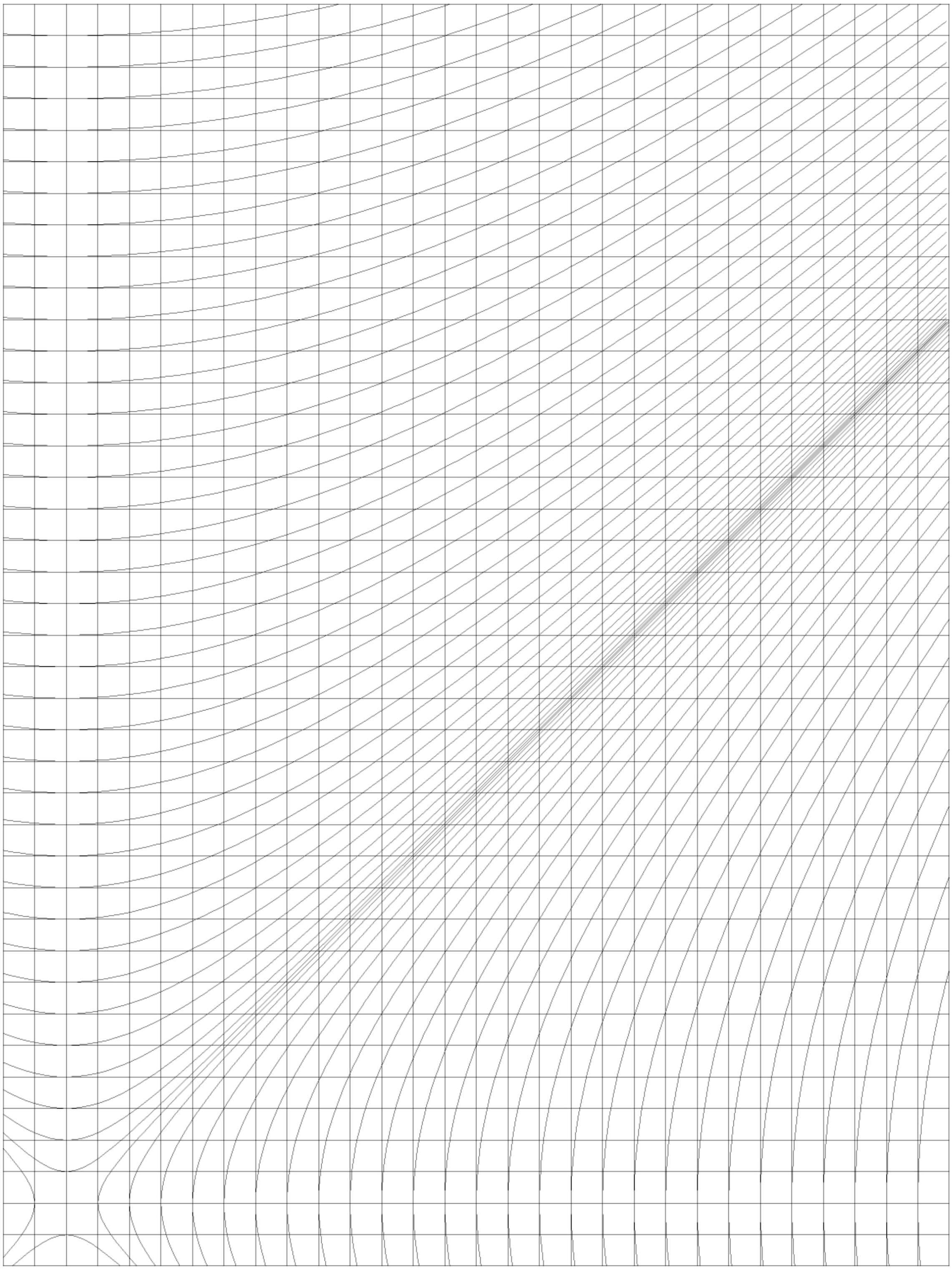

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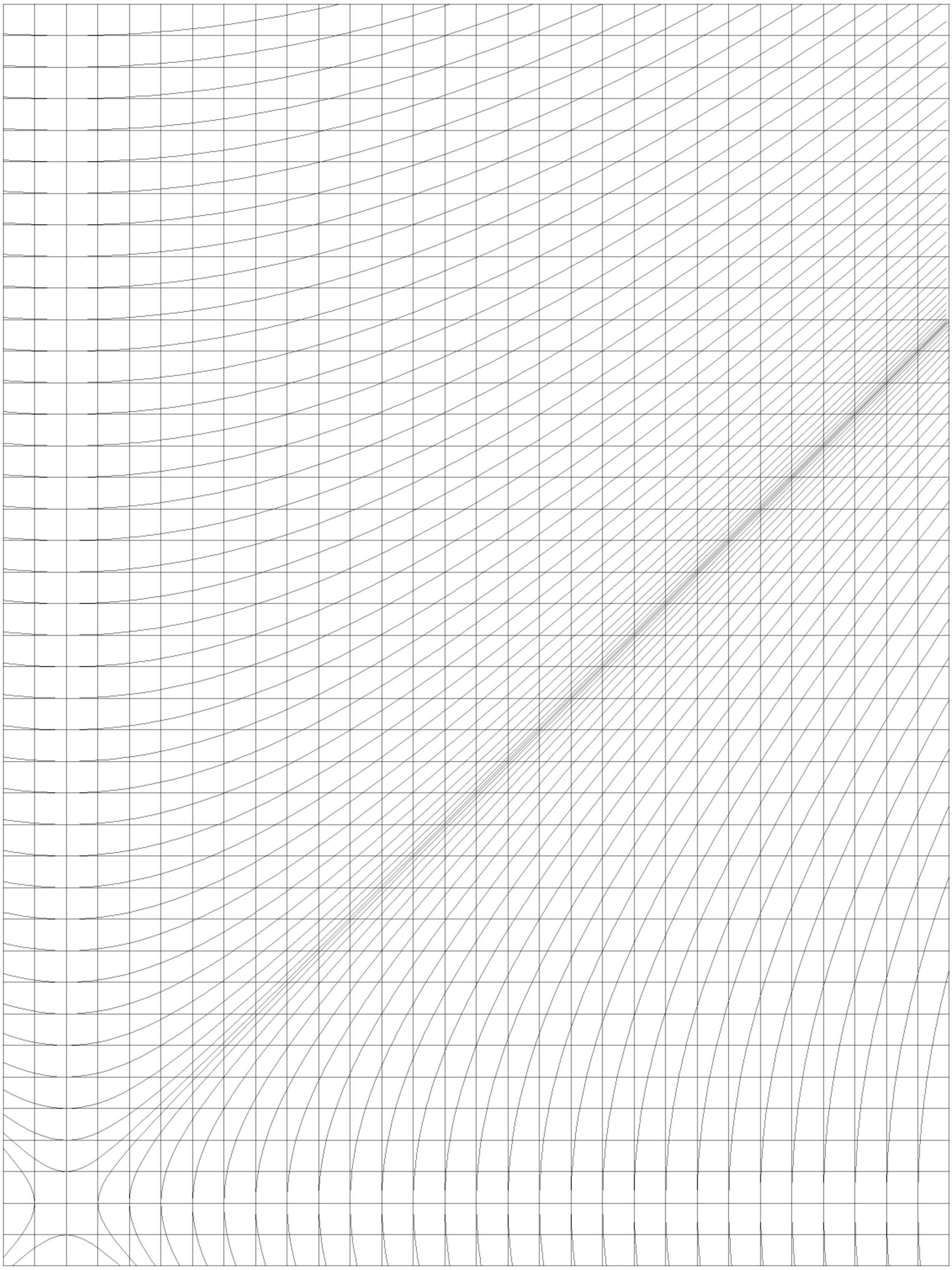

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