

Linear Algebra - Resit Exam

June 23, 2025 08.30-10.30

IMPORTANT:

At the end place your exam in the pile, **under the envelope** corresponding to your tutorial group

Do not cover the envelope with your exam, so that others can read the tutorial group number on the envelope

To avoid delays when placing the exams, make sure you **know your tutorial group number in advance** (see BS)

To avoid confusion in case exams get mixed up in the piles, write your name and student number on all pages

Number the pages of the exam sequentially

Exams not returned to the correct pile or pages without name/student number may not get graded

Exam rules:

- You can have a “cheat sheet”. This is an A4 paper written on one side (see above).
 - You are NOT allowed to have books, course notes, homework assignments, etc., laptops, e-readers, tablets, telephones, etc.
 - You can use a normal calculator (not a programmable/graphic one).
 - Give a clear explanation of your answer and show any relevant computations.
 - You get no points for a result without any calculation/explanation.
 - If you're asked explicitly to use a certain method, you'll get no points if you use a different one even if your answer is right.
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QUESTIONS:

1. 3 Given the matrix A

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix},$$

- (a) 1 Use the properties of partitioned matrices to calculate the matrix $(A^T)^2$.
- (b) 1 Use the properties of partitioned matrices to calculate the inverse of A .
- (c) 1 Use as few calculations as possible to determine whether the following matrix is invertible or not (you do not need to calculate the inverse):

$$A = \begin{bmatrix} 4 & 0 & 0 & -8 & 6 \\ -3 & 1 & -1 & 6 & -2 \\ 1 & 4 & -4 & -2 & 8 \\ 3 & -2 & 2 & -6 & 7 \\ -2 & 1 & -1 & 4 & -3 \end{bmatrix}.$$

2. 1 Given the vectors $\mathbf{v} = \begin{bmatrix} \beta \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -1 \\ \alpha \\ 2 \end{bmatrix}$, determine for which value(s) of α and β the vector $\mathbf{v}$ is in $\text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

3. 2 For a linear transformation $L: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ it is given that:

$$L\left(\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, L\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \text{ and } L\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

- (a) 1 Find the matrix A (defined by $L(\mathbf{x}) = A\mathbf{x}$) and show that $\begin{bmatrix} -9 \\ -2 \end{bmatrix}$ is the image of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
- (b) 1 Three 3D points are given by $P(2, 1, 3)$, $Q_1(1, 1, 1)$ and $Q_2(1, 1, 2)$. Determine and describe the images of the segment through P and Q_1 and segment through P and Q_2 .

Please turn over

4. 2 Check whether the following are eigenvalues and eigenvectors of a matrix. Justify your answer. If the answer is yes, give the matrix.

(a) 1 $\lambda_1 = \frac{5}{3}, \lambda_2 = 3, \lambda_3 = -1, \mathbf{v}_1 = \begin{bmatrix} 4 \\ 5 \\ -9 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 5 \\ 8 \end{bmatrix}.$

(b) 1 $\lambda_1 = -4, \lambda_2 = 2, \lambda_3 = 2, \lambda_4 = 1, \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -6 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$

Hint: Notice that, even if in this case you have a 4×4 matrix, it is a partitioned matrix

5. 2 Find the **real** solution of the following initial-value problem:

$$x'_1 = 2x_1 + x_2 + x_3$$

$$x'_2 = -2x_3$$

$$x'_3 = 2x_2$$

with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$

*Hint: You can use directly the formula in the book that gives linearly independent **real** solutions. But you still need to explain how you did the rest and why you use this formula.*

In case you need this, 2 of the 3 e-values and corresponding e-vectors are $\lambda_1 = 2, \lambda_2 = 2i, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -i \\ 2i \\ 2 \end{bmatrix}$

NOTE: Maximum points possible, $p = 10$. Grade, $g = 0.9p + 1$

Solutions

If not explicitly said, the students lose 0.1 points for each **new** algebraic mistake. If they do make a mistake, check the rest and if the result they get is self consistent given that error, give the remaining of the points.

1. (a) First $A^T = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 8 & 6 \end{bmatrix}$.

This is a block diagonal matrix, so I can operate on the blocks. I therefore need to calculate (I do not show the intermediate steps, just the final result, but the students have to):

They get 0 points if they do not use the blocks, even if the result is correct, because the question said they had to do that.

They get 0.4 points for the method.

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 12 & 31 \end{bmatrix},$$

$$\begin{bmatrix} 7 & 5 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 89 & 65 \\ 104 & 76 \end{bmatrix},$$

and the central block is 2, so $2^2 = 4$.

$$\text{Therefore } (A^T)^2 = \begin{bmatrix} 7 & 18 & 0 & 0 & 0 \\ 12 & 31 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 89 & 65 \\ 0 & 0 & 0 & 104 & 76 \end{bmatrix}.$$

They get 0.6 points for the solution

- (b) To calculate A^{-1} I can calculate the inverse of the blocks (I do not show the intermediate steps, just the final result, but the students have to):

They get 0 points if they do not use the blocks, even if the result is correct, because the question said they had to do that.

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -4 \\ -\frac{5}{2} & \frac{7}{2} \end{bmatrix},$$

and the central block is 2, so $2^{-1} = 1/2$.)

They get 0.4 points for all the three inverses

$$\text{Therefore } A^{-1} = \begin{bmatrix} -5 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -\frac{5}{2} & \frac{7}{2} \end{bmatrix}.$$

They get 0.6 points for the solution

- (c) Simply notice that column 4 is equal to column 1 times -2 , hence $\det A = 0$.

They get 0.6 points for noticing this

0.4 points for the result. They get the full points also if they calculate the full determinant, but make a note in this case pointing to the shorter solution

2. I need to see whether I can find numbers x_1, x_2, x_3 to write $\mathbf{v} = x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c}$:

0.3 points for this or similar statement; give the points if they do not make the statement explicitly but use it to solve this problem.

$$x_1 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ \alpha \\ 2 \end{bmatrix} = \begin{bmatrix} \beta \\ 3 \\ 3 \end{bmatrix}, \text{ or equivalently:}$$

Then write the augmented matrix and operate on the rows:

$$\left[\begin{array}{ccc|c} -3 & 1 & -1 & b \\ -2 & 4 & a & 3 \\ 2 & 1 & 2 & 3 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} -3 & 1 & -1 & b \\ 0 & 10/3 & (3a+2)/3 & (9-2b)/3 \\ 0 & 0 & (-a+2)/2 & (2b+3)/2 \end{array} \right],$$

Give 0.4 points for the row reduction.

If $a \neq 2$ there is 1 solution.

Give 0.1 points for the result.

If $a = 2$ and $b = -3/2$ the last row is all 0's, there is a free variable and so there are infinite solutions.

Give 0.1 points for the result.

If $a = 2$ and $b \neq -3/2$ the last row is inconsistent, so no solutions.

Give 0.1 points for the result.

Here it is important that they say (somehow) that both conditions (hence the “and”) must hold. They get 0 points if they do not say that either explicitly or implicitly (in the latter case it must be clear that they do so), or if they say “or”.

3. (a) I need to find the effect of the linear transformation of the unit vectors.

They get 0.3 points for this or similar statement, even if later they make algebraic mistakes

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -2\mathbf{e}_2, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = -\mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \Rightarrow$$

$$L(-2\mathbf{e}_2) = -2L(\mathbf{e}_2) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$L(-\mathbf{e}_1 + 2\mathbf{e}_2) = -L(\mathbf{e}_1) + 2L(\mathbf{e}_2) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$L(\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3) = L(\mathbf{e}_1) + 2L(\mathbf{e}_2) + 3L(\mathbf{e}_3) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

$$\text{From the first one } L(\mathbf{e}_2) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

0.2 points for the vector that is the transformation of this unit vector

$$\text{From the second one } L(\mathbf{e}_1) = -\begin{bmatrix} 6 \\ 0 \end{bmatrix} + 2L(\mathbf{e}_2) = -\begin{bmatrix} 6 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ -2 \end{bmatrix}$$

0.2 points for the vector that is the transformation of this unit vector

$$\text{From the third one } 3L(\mathbf{e}_3) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} - L(\mathbf{e}_1) - 2L(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -12 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 18 \\ 3 \end{bmatrix} \Rightarrow L(\mathbf{e}_3) = \begin{bmatrix} 6 \\ 1 \end{bmatrix}.$$

0.2 points for the vector that is the transformation of this unit vector

$$A = [L(\mathbf{e}_1) \ L(\mathbf{e}_2) \ L(\mathbf{e}_3)] = \begin{bmatrix} -12 & -3 & 6 \\ -2 & -1 & 1 \end{bmatrix}.$$

0.1 points for the final matrix.

$$(b) \text{ P: } L\left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -9 \\ -2 \end{bmatrix}, \text{ Q1: } L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -9 \\ -2 \end{bmatrix}, \text{ Q2: } L\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

Both P and Q1 are mapped onto $\begin{bmatrix} -9 \\ -2 \end{bmatrix}$; the segment from P to Q1 in 3D is projected onto the same point in 2D.

0.5 points for this result

The segment from P to Q2 in 3D is projected onto the segment from $(-9, -2)$ and $(-3, -1)$ in 2D.

0.5 points for this result

4. (a) I need to check whether the three vectors are linearly independent.

0.3 points for this or similar statement; they get the points also if they do not make the statement explicitly but use it

In this case it is easy to see that the vectors are not linearly independent because, if I call $P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$, $\det P = 0$.

0.4 points for calculating the determinant. They get the full 0.6 points also if they use a different, valid, method to prove that the vectors are not independent

Then P^{-1} does not exist, and I won't be able to calculate $A = PDP^{-1}$. So the answer is no, these are not the eigenvalues and eigenvectors of a matrix.

0.3 points for the answer

- (b) In this case the vectors are linearly independent. To see this I call $P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$.

Since this is a block diagonal matrix, it is easy to calculate the determinant of the two (2×2) blocks and

$$\text{multiply them by each other to get that } \det P = \det \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ -6 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix} = 6.$$

0.2 points for using the determinant to prove linear independence

0.3 points for the determinant

So the answer is yes, these are the eigenvalues and eigenvectors of a matrix.

To find this matrix, A, I need to invert P, so I can write $A = PDP^{-1}$ where D is a diagonal matrix with the eigenvalues in the diagonal. To invert P I take advantage from the fact that P is block diagonal. The blocks, and corresponding inverses are:

$$P_{11} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, P_{22} = \begin{bmatrix} -6 & 0 \\ 5 & 1 \end{bmatrix}, P_{11}^{-1} = \begin{bmatrix} -1/6 & 0 \\ 5/6 & 1 \end{bmatrix}, \text{ and } P_{22}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}, \text{ and hence:}$$

$$P^{-1} = \begin{bmatrix} 0 & 0 & -1/6 & 0 \\ 0 & 0 & 5/6 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}.$$

0.3 points for the inverse of P

Then:

$$A = PDP^{-1} = \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ -6 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1/6 & 0 \\ 0 & 0 & 5/6 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 5 & 2 \end{bmatrix}$$

0.2 points for A

5. The system can be written in matrix form as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

I first find the evals/evecs of A (I do not show the steps, but the students have to).

The evals are $\lambda_1 = 2$, $\lambda_2 = 2i$, $\lambda_3 = -2i$, and the corresponding events are: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -i \\ 2i \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} i \\ -2i \\ 2 \end{bmatrix}$.

0.9 for the 3 evals and 3 events; 0.1 points for each eval and 0.2 points for each event.

It is okay if they calculate only $\mathbf{v}_1$ and $\mathbf{v}_2$, and say that $\mathbf{v}_3$ must be the complex conjugate of $\mathbf{v}_2$.

At this point they may go as I do below, or they can decide to change variables to decouple the equations as I explained in the class (see my notes). Both ways are fine. if they choose to decouple the equations, at the end they **must** transform back to the original variables.

I write $\lambda_2 = a + ib$, where $a = 0$ and $b = 2$.

The general **real** solution will then be:

They are allowed to take the formula directly from the book.

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 [\operatorname{Re}(\mathbf{v}_2) \cos bt - \operatorname{Im}(\mathbf{v}_2) \sin bt] e^{at} + c_3 [\operatorname{Re}(\mathbf{v}_2) \sin bt + \operatorname{Im}(\mathbf{v}_2) \cos bt] e^{at}.$$

It is also possible to write the general solution $\mathbf{x} = \sum_{i=1}^3 c_i \mathbf{v}_i e^{\lambda_i t}$, where λ_i and $\mathbf{v}_i$ are possibly complex, and keep only the real part of the solution. In all cases the results should be the same, but some of the intermediate steps may differ from what I have below.

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \cos 2t - \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \sin 2t \right) + c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \sin 2t + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \cos 2t \right) = \begin{bmatrix} c_1 e^{2t} + c_2 \sin 2t - c_3 \cos 2t \\ -2c_2 \sin 2t + 2c_3 \cos 2t \\ 2c_2 \cos 2t + 2c_3 \sin 2t \end{bmatrix}.$$

0.6 points for the general solution.

Applying the initial condition:

$$\begin{bmatrix} c_1 - c_3 \\ 2c_3 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

I can solve this to find $c_1 = 3$, $c_2 = 1$, $c_3 = 1$. The solution of the initial-value problem is then:

0.3 points for the c_i , 0.1 points for each.

$$\mathbf{x}(t) = \begin{bmatrix} 3e^{2t} + \sin 2t - \cos 2t \\ -2 \sin 2t + 2 \cos 2t \\ 2 \cos 2t + 2 \sin 2t \end{bmatrix}.$$

0.2 points for the solution of the initial-value problem.