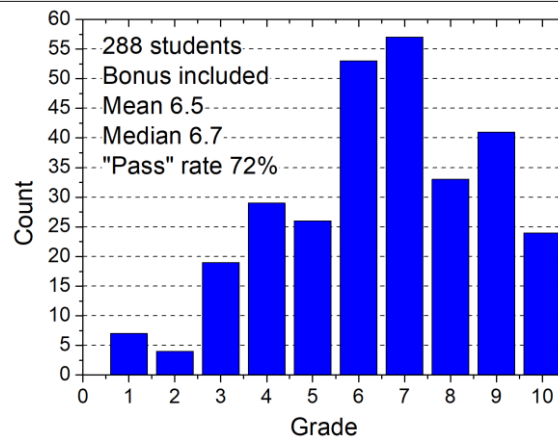


Electricity and Magnetism, Test 1 Feb 20 2024 18:30-20:30

4 problems, 34 points

Write your name and student number on each answer sheet. Use of a calculator is allowed. You may make use of one A4 (double sided) with handwritten notes and of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\vec{A}$ is a vector, $\hat{x}$ is the unit vector into the x -direction. **In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol.**

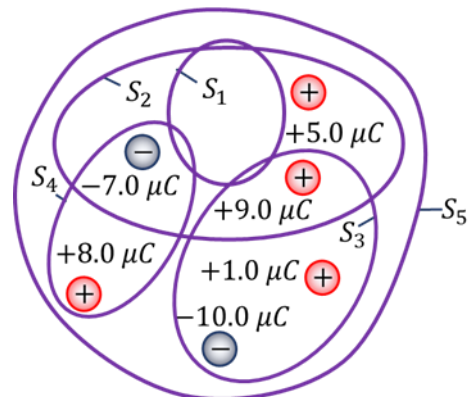


Problem 1. Conceptual questions (8 points)

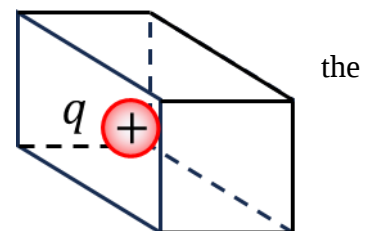
1. Explain whether each of the following statements is true or false: (3 points)

- A. Adding a constant to the potential changes the electric field
- B. The potential can vary with z even if the z -component of E is constant
- C. The electric field points in the direction of steepest increase of the potential

2. The figure shows six point charges that all lie in the same plane. It also shows five Gaussian surfaces $S_1 - S_5$, which are closed above and below the paper. Rank the surfaces in order of electric field flux through them, from most positive to most negative, and explain your reasoning. (3 points)



3. A charge is placed inside a box (see the figure). If all dimensions of the box are increased by a factor of 3, how will electric flux through the box change? (2 points)



Answers: (scored points 74%)

1. 1 point for each correct answer

A. False: $\vec{E} = -\vec{\nabla}V$ so adding a constant to V doesn't change a thing

B. True: $V(\vec{r}) \equiv - \int_0^{\vec{r}} \vec{E} \cdot d\vec{l} = -E(=const) \int_0^{\vec{r}} dz$

C. False: $\vec{E} = -\vec{\nabla}V$ so the minus sign indicates the direction opposite of the steepest increase

2. The flux through a closed surface is proportional to the charge enclosed within the surface.

$S_1: 0; S_2: +7; S_3: 0; S_4: +1; S_5: +6$ (2 points)

Therefore, $S_2 - S_5 - S_4 - (S_3 = S_1)$ (1 point)

3. Each part of the surface of the box will be three times farther from the charge so the electric field will decrease by a factor of 9. But the area of the box will increase by the same factor of 9. So the flux will be unchanged. (2 points)

Typical mistakes:

(1) Mixing up divergence and gradient

(1B): Explaining how $\vec{E}$ varies rather than how the potential varies

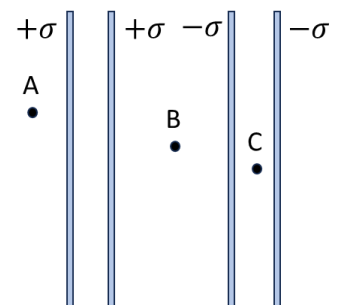
(1C): Forgetting about the minus sign when considering the direction of $\vec{E}$ in terms of the gradient of the potential

(2) Forgetting that the electric field magnitude changes when considering the flux through the new surface

Problem 2. Charged planes (8 points)

1. Derive the electric field of an infinite plane with a uniform positive charge density σ . (4 points)

2. Consider the situation with four infinite charged planes shown in the figure on the right. Rank the points A, B, and C in order of small to large electric field magnitude. Prove your answer with calculations (4 points)



Answers: (scored points 70%)

1. Follow Example 2.5 from Griffiths; lecture notes will also do. Essential elements which the learner should demonstrate, are:

- How to choose the Gaussian surface (the pillbox)

(1 point)

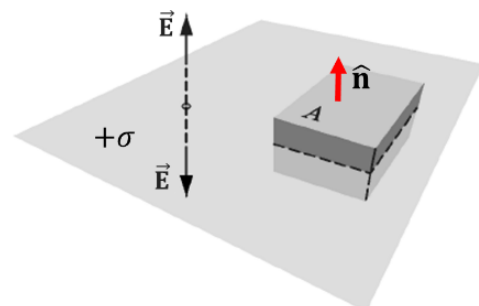
- How to apply Gauss's law (2 points)

$$\oint_S \vec{E} \cdot d\vec{a} = |\vec{E}| \oint_S da = 2A|\vec{E}|$$

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}; Q_{enc} = \sigma A$$

- The final result $\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n}$ (1 point)

2. The superposition principle will be used (1 point)



$$\vec{E}(A) = 0 \left(-(+\sigma + \sigma) - (-\sigma - \sigma) \right) \text{ (1 point)}$$

$$\vec{E}(B) = \frac{2\sigma}{\epsilon_0} \hat{n} \left(+(+\sigma + \sigma) - (-\sigma - \sigma) \right); \hat{n} \text{ directed to the right} \text{ (1 point)}$$

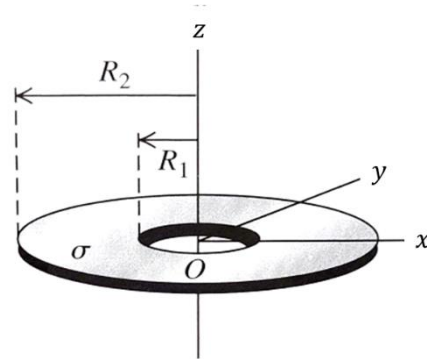
$$\vec{E}(C) = \frac{\sigma}{\epsilon_0} \hat{n} \left(+(+\sigma + \sigma) - \sigma - (-\sigma) \right) \text{ (1 point)}$$

Typical mistake:

Adding vectors up wrongly

Problem 3. Electric field (10 points)

A thin disk with radius R_2 has a round hole in its center with radius R_1 , as shown in the figure. The disk has a uniform positive surface charge density σ on its surface. In this problem, we consider **only** points along the line through the center of the arrangement (the axis labeled 'z' in the figure).



1. Sketch qualitatively the z-component of the electric field as a function of z. Include both negative and positive z on your horizontal axis, and briefly explain the value at $z = 0$ and z-dependence at $z \rightarrow \infty$ (4 points)

2. Calculate the electric field at an arbitrary point on the z axis. (6 points)

You may use this integral (but may not need to):

$$\int \frac{z}{(z^2 + a^2)^{3/2}} dz = -\frac{1}{\sqrt{z^2 + a^2}}$$

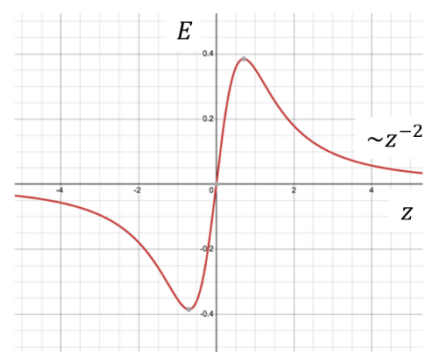
Answers: (scored points 53%)

1. $\vec{E}(z = 0) = 0$. It makes sense as the in-plane component is zero and we sit in the plane so there cannot be any other components (1 point)

At large distances the configuration of the charged object doesn't matter (it becomes a point object) so the answer is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 z^2} \hat{z} \text{ with } Q = \sigma\pi(R_2^2 - R_1^2) \text{ (1 point)}$$

2 points for the correct graph



2. From the symmetry of the problem it is clear that the electric field has only a component along z as the in-plane components will be compensated by the same charge from a symmetric position with respect to the hole axis. Therefore, we need to calculate only the z -component (1 point)

Note: if a learner calculates the xy -component directly, it also counts as 1 point

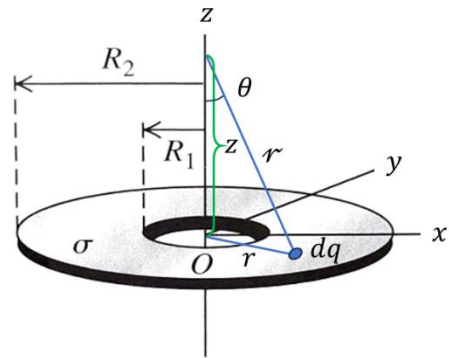
We use cylindrical system of coordinates: (1 point for the correct model)

$$r^2 = r^2 + z^2; \cos\theta = \frac{z}{\sqrt{r^2 + z^2}}$$

$$\begin{aligned} \vec{E} &= \frac{\hat{z}}{4\pi\epsilon_0} \int \frac{\sigma}{r^2 + z^2} \cos\theta \, r \, dr \, d\phi = \frac{\hat{z}}{4\pi\epsilon_0} \int_{R_1}^{R_2} \frac{\sigma}{r^2 + z^2} \frac{z}{\sqrt{r^2 + z^2}} r \, dr \int_0^{2\pi} d\phi \\ &= \frac{\sigma z \hat{z}}{4\pi\epsilon_0} 2\pi \int_{R_1}^{R_2} \frac{r}{(r^2 + z^2)^{\frac{3}{2}}} \, dr = \frac{\sigma z \hat{z}}{4\pi\epsilon_0} 2\pi \left(-\frac{1}{\sqrt{r^2 + z^2}} \right) \Big|_{R_1}^{R_2} \\ &= \frac{2\pi\sigma}{4\pi\epsilon_0} z \left(\frac{1}{\sqrt{R_1^2 + z^2}} - \frac{1}{\sqrt{R_2^2 + z^2}} \right) \hat{z} \quad (4 \text{ points}) \end{aligned}$$

If the direction of $\vec{E}$ is not specified (either as $\hat{z}$ or in words), -1 point

Note that there is an alternative approach when the disk is treated as a combination of infinitely thin circle loops. It is also correct if solved properly (so the same amount of points)

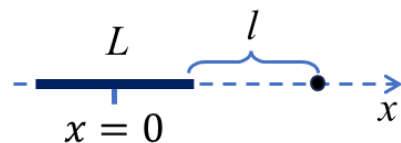


Typical mistakes:

1. Problems with defining infinitesimal area element for a disk
2. Confusing electric field and electric field lines
3. Did not integrate over $d\phi$, just dr ; so there was an extra 2π on the denominator of the final answer
4. Some students did not invoke symmetry (neither in first or second part) to justify why we need to consider only the z -component.

Problem 4. Potential (8 points)

A straight thin wire of length L carries a linear charge density λ . Assume the wire is centered at the origin and parallel to the x -axis.



1. Find a potential at the axis of the wire at a distance l from the end of the wire (4 points)
2. What potential do you expect at long distances from the wire? (you don't need to solve (1) for this) (2 points)

Tip: you might find $\ln(1 + L/l) \approx L/l$ useful

3. Give an expression for the three-dimensional volume charge density ρ in this problem. (2 points)

Tip: use Dirac's delta-function

Answers : (scored points 38%)

$$\begin{aligned} 1. V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\lambda}{r} dx = \frac{\lambda}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{dx}{l + L/2 - x} = -\frac{\lambda}{4\pi\epsilon_0} \ln \left(l + \frac{L}{2} - x \right) \Big|_{-L/2}^{L/2} \\ &= \frac{\lambda}{4\pi\epsilon_0} \ln \frac{l+L}{l} = \frac{\lambda}{4\pi\epsilon_0} \ln \left(1 + \frac{L}{l} \right) \quad (4 \text{ points}) \end{aligned}$$

2. If $l \gg L$,

$$V(\vec{r}) = \frac{\lambda}{4\pi\epsilon_0} \ln(1 + L/l) \approx \frac{\lambda}{4\pi\epsilon_0} \frac{L}{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{l} \text{ as expected for a point charge } (2 \text{ points})$$

Note that the right answer could be given without the Taylor expansion. It still deserves **2 points**

3. $\rho = \lambda\delta(y)\delta(z)$ for $|x| \leq L/2$, and zero otherwise **(2 points)**

Alternatively, $\rho = \lambda(\theta(x + L/2) - \theta(x - L/2))\delta(y)\delta(z)$, where $\theta(x)$ the step function

Typical mistakes:

1. Some students tried to find the potential via the electric field by using Gauss's law, which only works for infinite wires because of symmetry in the surface integral of $\vec{E}$.
2. Some students mixed up two different definitions of the potential.
3. Some students forgot that at large distances electric sources begin to behave as point source.
4. Many students used the dirac delta function over all coordinates, including the x component, which is the case for point charges.