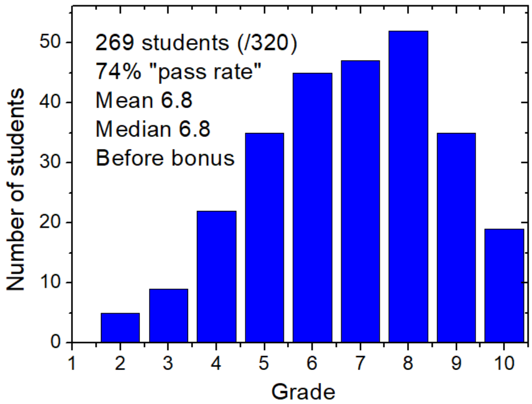
Electricity and Magnetism, Test 1 Feb 21 2025 18:30-20:30

4 problems, 34 points (and 2 bonus points)

Write your name and student number on each answer sheet. Use of a calculator is allowed. You may make use of one A4 (double sided) with handwritten notes and of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\vec{\mathbf{A}}$ is a vector, $\hat{\mathbf{x}}$ is the unit vector into the x-direction. **In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol**.



Problem 1. Conceptual questions (10 points; 2 points each) Average 6.6 (66%)

In all cases, explain your answers (if needed, with calculations)

A. A negative point charge moves along a circular orbit around a positive point charge. What aspect(s) of the electric force on the negative point charge will remain constant in the Cartesian coordinates as it moves?

(i). Magnitude
(ii). Direction
(iii). Both magnitude and direction
(iv). Neither magnitude nor direction

Answer: (i) the magnitude remains constant (the separation remains the same) but the direction changes (directed towards the positive charge)

Typical mistakes: not reading the part about using cartesian coordinates, and assuming spherical

B. Is this a possible electrostatic field (k is a nonzero constant)?

$\vec{\mathbf{E}} = k(xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}})$

Answer: No because $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} \neq 0$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(3xz) - \frac{\partial}{\partial z}(2yz)\right] + \cdots = \hat{\mathbf{x}}[0 - 2y] \neq 0$$

Typical mistakes: Calculated divergence instead of curl.

C. For what kind of surface is Gauss's law valid?

(i) Any closed surface
(ii) Closed surfaces that match the symmetry of the problem
(iii) Any kind of surface

Answer: (i) Gauss's law does not impose any restrictions on the shape of the *closed* surface. Choosing a surface that matches the symmetry of the problem (ii), is a matter of convenience for calculations, not a limitation of Gauss's law. The surface must be *closed*, which excludes (iii).

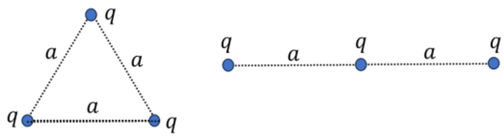
Typical mistakes: Thinking symmetry is required

D. Electric field is a vector quantity while the potential is a scalar. How come that a scalar contains all the information about a vector?

Answer: because $\vec{\mathbf{E}}$ is a conservative field ($\vec{\nabla} \times \vec{\mathbf{E}} = 0$) so its components are not independent.

Typical mistakes: Just giving relation between $\vec{\mathbf{E}}$ is the gradient of V.

E. Three identical point charges q are arranged in two different configurations: (i) at the vertices of an equilateral triangle with side length a, and (ii) in a straight line, with each charge separated by a distance a (see the figure). Which configuration has a higher electrostatic potential energy?



Answer:

(i): $W = \dfrac{1}{8\pi\epsilon_0}\dfrac{3q^2}{a}$

(ii): $W = \dfrac{1}{8\pi\epsilon_0}\left(\dfrac{2q^2}{a} + \dfrac{q^2}{2a}\right) = \dfrac{1}{8\pi\epsilon_0}\dfrac{2.5q^2}{a}$

The energy is higher in the first configuration.

An alternative answer: In the first configuration (triangle), the distance between each charge is shorter than in the second configuration (line). According to the equation for work, this suggests that the energy is higher in the first configuration.

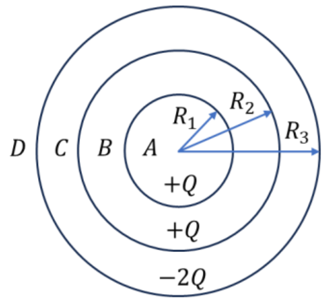
Typical mistakes:

(i) Mixing potential and electrostatic potential energy (even though we explicitly discussed the difference during lectures)

(ii) Counting the lines between the charges as the reasoning for the triangle having higher energy (i.e. there are 3 lines in a triangle compared to 2 on a straight line)

Problem 2. Charged spheres (6 points) Average 4 (67%)

You have a system of 3 concentric spherical shells with radii R_1, R_2 and R_3, each of which carries uniformly-distributed charges $+Q$, $+Q$ and $-2Q$, respectively (see the figure). Find the electric field in the four regions marked as A, B, C and D.



Answers:

Because of spherical symmetry, we choose Gaussian's surfaces as shells with their centra at the origin

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{1}{\epsilon_0} Q_{enc} \quad \text{(2 points)}$$

Region A: $Q_{enc} = 0, \vec{\mathbf{E}} = 0$ (1 point)

Region B: $E\ 4\pi r^2 = Q_{enc} = +Q, \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \dfrac{1}{4\pi\epsilon_o}\dfrac{Q}{r^2}\hat{\mathbf{r}}$ (1 point)

Region C: $E\ 4\pi r^2 = Q_{enc} = +2Q, \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \dfrac{1}{4\pi\epsilon_o}\dfrac{2Q}{r^2}\hat{\mathbf{r}}$ (1 point)

Region D: $Q_{enc} = 0, \vec{\mathbf{E}} = 0$ (1 point)

Typical mistakes:

(i) Not writing out the Gauss law or using the wrong Gaussian surfaces.

(ii) Not realizing Gaussian surface has variable r and using the radius of the circles R_1, R_2, and R_3 as the distance variable instead of the correct r

(iii) Forgetting the direction vector $\hat{\mathbf{r}}$.

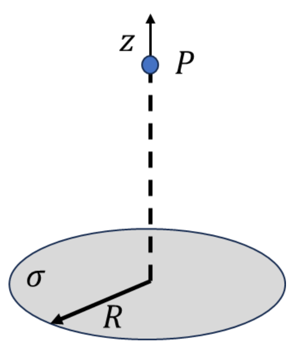
(iv) Using infinite plane arguments to get the electric field pointing both in and out of the shells

(v) Not realizing that in D, the charges cancel out perfectly

Problem 3. Electric field of a charged object (10 points + 2 bonus)

Average 6.6 (66% bonus excluded)

A flat, infinitely thin circular disk of radius R (see the figure) carries a uniform positive surface charge density σ.



1. Sketch qualitatively the z-component of the electric field as a function of z above the center of the disk. Include both negative and positive z on your horizontal axis, and briefly explain z-dependences at $z \ll R$ and $z \gg R$. **(4 points)**

Tip: be careful at $z = 0$

2. Now directly calculate the electric field at a distance z above the center of the disk. **(4 points)**

Tip: You may use this integral (but may not need to):

$$\int \frac{r}{(r^2+z^2)^{3/2}}\,dr = -\frac{1}{\sqrt{r^2+z^2}} + const$$

3. Give an expression for the three-dimensional volume charge density ρ in this problem. **(2 points)**

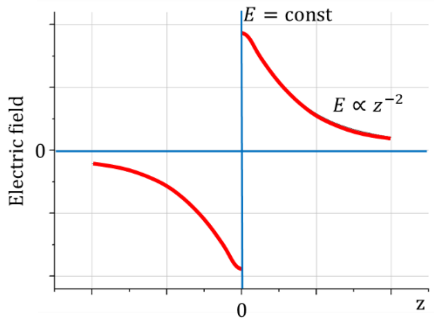
Tip: use Dirac's delta-function

4. **Bonus question:** What does the formula calculated in (2), give in the limit $z \gg R$ (but not $z \to \infty$ which is trivial)? Explain why it makes sense. **(2 points)**

Tip: you may find the following expression useful: $\sqrt{1+x} \cong 1 + x/2$ for $x \ll 1$

Answers:

1. The electric field component E_z should be positive for $z > 0$ and negative for $z < 0$, indicating that the field points away from the disk for a positively charged surface. (2 points)



For $z \ll R$, the disk appears infinitely large so that the field should behave similarly to that of an infinite plane of charge:

$$\vec{\mathbf{E}}(z \ll R) = \frac{\sigma}{2\epsilon_o}\hat{\mathbf{z}} = \text{const} \quad \text{(1 point)}$$

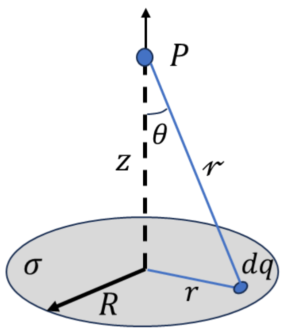
At large distances, the specific configuration of a charged object with the finite size becomes irrelevant, as it effectively behaves like a point charge. Therefore, the behaviour is

$$\vec{\mathbf{E}} = \frac{\sigma\pi R^2}{4\pi\epsilon_o z^2}\hat{\mathbf{z}} \propto \frac{1}{z^2}\hat{\mathbf{z}} \qquad \text{(1 point)}$$

2. From the symmetry of the problem it is clear that the electric field has only a component along z as the in-plane components will be compensated by the same charge from a symmetric position with respect to the disk axis. Therefore, we need to calculate only the z-component (1 point)

Note: if a learner calculates the r-component directly, it also counts as 1 point

We use the cylindrical system of coordinates: (1 point for the correct model)



$$\mathscr{r}^2 = r^2 + z^2;\ cos\theta = \frac{z}{\sqrt{r^2+z^2}}$$

$$\vec{\mathbf{E}} = \frac{\hat{\mathbf{z}}}{4\pi\epsilon_o}\int \frac{\sigma}{r^2+z^2} cos\theta\ r\,dr\,d\varphi$$

$$= \frac{\hat{\mathbf{z}}}{4\pi\epsilon_o}\int_0^R \frac{\sigma}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}} r\,dr \int_0^{2\pi} d\varphi$$

$$= \frac{\sigma z\,\hat{\mathbf{z}}}{4\pi\epsilon_o} 2\pi \int_0^R \frac{r}{(r^2+z^2)^{\frac{3}{2}}}\, dr = \frac{\sigma z\,\hat{\mathbf{z}}}{4\pi\epsilon_o} 2\pi \left(-\frac{1}{\sqrt{r^2+z^2}} \right)\Big|_0^R$$

$$= \frac{2\pi\sigma}{4\pi\epsilon_o} z \left(\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}} \right) \hat{\mathbf{z}} = \frac{\sigma}{2\epsilon_o} \left(\mathrm{sign}(z) - \frac{z}{\sqrt{R^2+z^2}} \right) \hat{\mathbf{z}}$$ (4 points)

Absence of the modulus around z does not result in any penalty if the direction of $\vec{\mathbf{E}}$ is specified, either on graph or in words. The last step in simplification is not compulsory either.

If the direction of $\vec{\mathbf{E}}$ is not specified (either as $\hat{\mathbf{z}}$ or in words or in graph), -1 point

An alternative approach is if the learner breaks the disk into rings of radius r and width dr, and uses a known expression (calculated at tutorials) for the field of the ring

$$\vec{\mathbf{E}}_{ring} = \frac{1}{4\pi\epsilon_o} \frac{z}{(r^2+z^2)^{3/2}} \sigma 2\pi r dr\ (= dQ)\hat{\mathbf{z}}$$

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_o} 2\pi\sigma z \int_0^R \frac{r}{(r^2+z^2)^{3/2}} dr\,\hat{\mathbf{z}} = \frac{\sigma}{2\epsilon_o} z \left[\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}} \right] \hat{\mathbf{z}}$$

The same number of points should be awarded as before

3. $\rho(r,z) = \begin{cases} \sigma\,\delta(z) \text{ for } r < R \\ 0 \qquad\quad \text{for } r < R \end{cases}$ (2 points)

Or using the Heaviside step function, $\rho = \sigma\theta(R-r)\delta(z)$

4. $|\vec{\mathbf{E}}| = \frac{\sigma}{2\epsilon_o}\left[1 - \frac{1}{\sqrt{(R/z)^2+1}}\right] \cong \frac{\sigma}{2\epsilon_o}\left[1 - \left(1 - \frac{1}{2}\left(\frac{R}{z}\right)^2\right)\right] = \frac{\sigma}{4\epsilon_o}\frac{R^2}{z^2} = \frac{1}{4\pi\epsilon_o}\frac{\sigma\pi R^2}{z^2} =$

$= \frac{1}{4\pi\epsilon_o}\frac{Q}{z^2}$, where $Q = \sigma\pi R^2$ (2 bonus points)

i.e. exactly Coulomb's law as it should be

(for the bonus, the expansion of the field must be carried out explicitly; just writing Coulomb's law does not qualify)

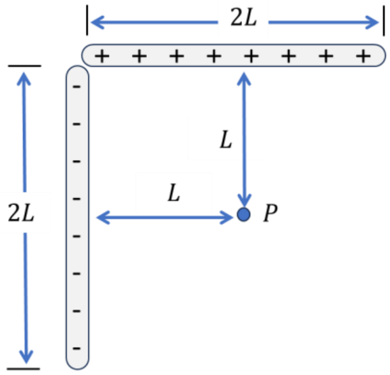
Typical mistakes:

(i) Many students thought the electric field would go to 0 at z=0 (as it was the case in the last-year test 1) -- even though it was explicitly mentioned in the tip, and I, noticing many wrong answers during the test, made an announcement about it

(ii) Forgetting the cosine in (2) or forgetting how to evaluate the integral at the bounds (many forgot to plug in $r = 0$ – although it might help raise some concerns about the correctness of the answer to the first sub-question).

Problem 4. Potential (8 points) Average 6.6 (60%)

Two infinitely thin, nonconducting straight rods with length $2L$ meet at a right angle (but don't touch each other; see the figure). The rods carry charges $+Q$ and $-Q$ distributed uniformly along their lengths. Your observation point P is situated at the distance L from each rod as shown.



1. Find the potential one rod produces at point P (6 points)

Tip: you may find the following integral useful:

$$\int \frac{1}{(L^2+x^2)^{1/2}}dx = \ln\left(x+\sqrt{L^2+x^2}\right) + const$$

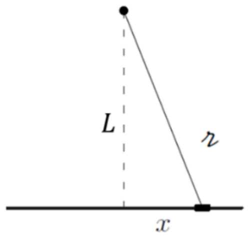
2. Find the potential both rods produce at point P (2 points)

Note that you do not need to know the exact answer to (1) to answer (2)

Answers:

1. This is similar to Problem 2.25 (2.26) considered at tutorials (6 points)

$$\text{For the positively} - \text{charged rod}, \lambda = \frac{Q}{2L}$$



$$V_+ = \frac{1}{4\pi\epsilon_o}\int_{-L}^{L}\frac{\lambda}{\sqrt{L^2+x^2}}dx = \frac{\lambda}{4\pi\epsilon_o}\left[\ln\left(x+\sqrt{L^2+x^2}\right)\right]\Big|_{-L}^{L}$$

$$= \frac{\lambda}{4\pi\epsilon_o}\left[\ln\left(L+\sqrt{L^2+L^2}\right) - \ln\left(-L+\sqrt{L^2+L^2}\right)\right]$$

$$= \frac{\lambda}{4\pi\epsilon_o}\ln\left(\frac{1+\sqrt{2}}{-1+\sqrt{2}}\right) = \frac{\lambda}{4\pi\epsilon_o}\ln\left(\frac{\left(1+\sqrt{2}\right)^2}{\left(-1+\sqrt{2}\right)\left(1+\sqrt{2}\right)}\right) = \frac{\lambda}{2\pi\epsilon_o}\ln\left(\frac{1+\sqrt{2}}{2-1}\right)$$

$$V_+ = \frac{1}{2\pi\epsilon_o}\frac{Q}{2L}\ln\left(1+\sqrt{2}\right)$$

Notice the neat simplification - it's always a nice touch! No points deducted if the learner chooses not to apply it.

2. The total potential is the sum of the two potentials which are of the same magnitude but opposite signs:

$V = V_+ + V_- = 0$ (2 points)

Without knowing the answer to #1: the system exhibits symmetry with respect to the line that starts at the intersection of the rods and passes through point P. Consequently, the potentials generated by each rod are equal in magnitude but opposite in sign. Since potentials are additive, the total result is zero. (the same 2 points)

Typical mistakes:

(i) Students thought the potential is a vector

(ii) Some first found the electric field and then integrated it to find the potential but all the integrals were incorrect

(iii) Some in order to find the field, used Gauss law despite the lack of symmetry at the ends of the rod

(iv) Some did not define the value of the line charge density so they just used λ even though that was not given in the problem formulation

(v) Some students integrated from 0 to 2L without realising that if the origin is at the end of the rod, the potential takes a different expression than the one needed to use the hint for the integral