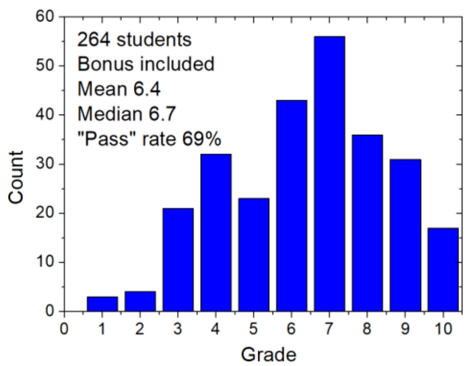
Electricity and Magnetism, Test 2 March 12 2024 18:30-20:30

4 problems, 40 points

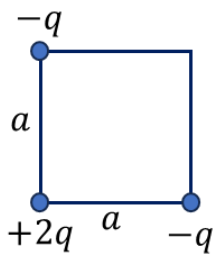
Write your name and student number on **each answer sheet**. Use of a calculator is allowed. You may make use of one A4 (double sided) with handwritten notes and of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\vec{\mathbf{A}}$ is a vector, $\hat{\mathbf{x}}$ is the unit vector into the x-direction. **In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol**.



Problem 1. Conceptual questions (10 points; 2 points for each)

In all cases, explain your answers! You don't need to calculate the exact values; simply provide your reasoning

A. Three charges are situated at the corners of a square with side a (see the figure). Is the work to bring in another charge, $+q$, from far away and place it in the fourth corner, positive or negative?



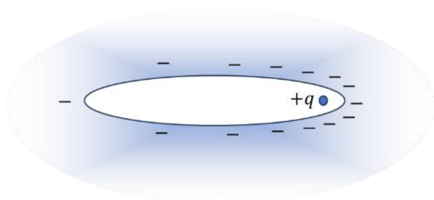
Answer: (similar to 2.31 from tutorials)

$$W \propto +q \sum V(\vec{r}) \propto -\frac{1}{a} - \frac{1}{a} + \frac{2}{a\sqrt{2}} \propto -2 + \frac{2}{\sqrt{2}} < 0$$

B. A point charge q is placed inside a cavity in an uncharged conductor, not touching the conductor. Is the force on q necessarily zero?

Answer: (Problem 2.40) No. For example, if it is very close to the wall, it will induce charge of the opposite sign on the wall, and it will be attracted to that wall.

Consider for example a long cavity in a conductor where the charge is placed near one end. The density of induced charges in the conductor will be the highest near that end while at the opposite end, it will be very low. On the other hand, the distance to those induced charges will also be longer than at the nearest end. As a result, the charge will be attracted to the nearest end.



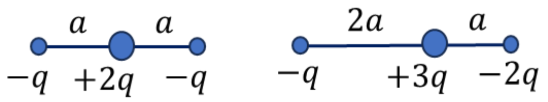
For a spherical cavity of radius R with a charge Q displaced from the center of the cavity by distance d, there is an exact solution for the force:

$$F = \frac{kQ^2}{R^2} \frac{\delta}{(\delta^2 - 1)^2}$$

where $\delta \equiv d/R$. Only if the charge is placed in the center of the cavity (i.e. $d = 0$), $F = 0$. (https://www.quora.com/What-is-the-force-on-charges-that-are-in-a-cavity-of-a-hollow-conducting-sphere)

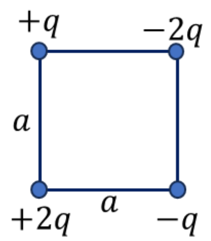
C. Sketch an arrangement of three non-overlapping point charges that has no dipole moment. Each charge may have any sign and magnitude you wish other than zero.

Answers (examples; learners might be more inventive):

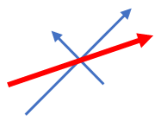


Note that you could take 3 equal charges and put the origin at the center but this needs to be specified explicitly.

D. Does the potential of this configuration fall off as r^{-2} at large distances r from the configuration?



Answer: this combination of charges is equivalent to two dipoles $(-q, +q)$ and $(-2q, +2q)$. Their combined dipole moment is non-zero (shown by a thick arrow) so YES, the potential fall off as r^{-2}



E. Explain briefly how a material with no intrinsic dipoles, ions, or free electrons can still become polarized when placed in an electric field. The field is too weak to ionize the material.

Answer: The nucleus in the electric field are pushed in the direction of the field, the electrons are pushed in the opposite direction. Their centra do not coincide any longer which results in a dipole, i.e. the material becomes polarized.

<u>Typical mistakes:</u>

For part (b), some students thought that because the electric field inside the conductor is zero then there would be no force exerted on the charge in the cavity

For part (c), configurations of three charges with net charge not equal to zero were proposed but no specification of the origin was made

Some students in part (d) talked about polar molecules when question specifically specified to consider a case with no intrinsic dipoles.

Problem 2 (8 points)
Find the energy stored in a uniformly charged solid sphere of radius R and charge q using

$$W = \frac{\epsilon_0}{2}\int_{All\ space} E^2\, d\tau$$

<u>Tip</u>: don't forget that there is the field inside and outside the sphere

Answer: (Problem 2.34b; 2.34a was solved in tutorials)
Electric field is found from Gauss's law:

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\begin{cases}\dfrac{\hat{\mathbf{r}}}{r^2} \text{ for } r > R \text{ (2 points)} \\ \dfrac{r\hat{\mathbf{r}}}{R^3} \text{ for } r < R \text{ (2 points)}\end{cases}$$

MP: I understand some students simply reproduced these from their cheatsheets. I recommend awarding half-points nevertheless – say, for recognizing important formula.

$$W = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\left\{\int_R^\infty \frac{1}{r^4} r^2 sin\theta\, d\theta d\varphi + \int_0^R \left(\frac{r}{R^3}\right)^2 r^2 sin\theta\, d\theta d\varphi\right\}$$

$$= \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\left\{\frac{1}{R}4\pi + \frac{1}{R^6}\frac{R^5}{5}4\pi\right\} = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{R} + \frac{1}{5R}\right) = \frac{3q^2}{20\pi\epsilon_0 R} \text{ (4 points)}$$

Typical mistakes:

- Some students interpreted the problem as one involving a conductor or a dielectric material, leading to an expression involving a zero electric field inside the solid sphere.
- Mistakes in integration involving a volume element in spherical coordinates.
- Missing direction (unit) vectors in the expressions of the electric field.

Problem 3 (10 points)
Consider a very long thick metal cylinder with inner radius a and outer radius b. We place a thin wire with line charge density λ all along the central axis.
A. Find the induced surface charge density at both surfaces of the metal cylinder
B. Sketch the magnitude of the electric field as a function of the radial distance s from the central axis, ranging from $s = 0$ to $s > b$, with $s = a$ and $s = b$ clearly marked.

Answer:
A. The charge of the wire of length is λl. The same charge in magnitude but of the opposite sign, should be induced at the inner surface of the metal because Gauss's law:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{1}{\epsilon_0}Q_{enc};\ \vec{\mathbf{E}} = 0 \text{ inside metal so } Q_{enc} = 0 \text{ (2 points)}$$

Therefore,

$$\sigma_{inner} = -\frac{\lambda l}{2\pi a l} = -\frac{\lambda}{2\pi a} \text{ (2 points)}$$

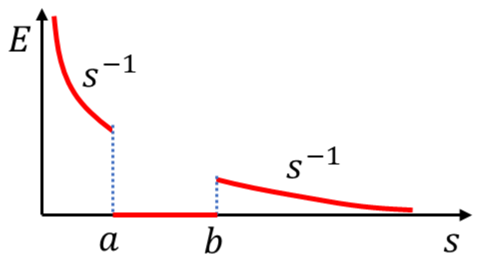
The charge per unit length at the outer surface should be equal to the charge at the inner surface, i.e.

$$\sigma_{outer} = \frac{\lambda}{2\pi b} \text{ (2 points)}$$

B. The field inside is readily found from Gauss's law:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{1}{\epsilon_0}Q_{enc}; E\ 2\pi s l = (\lambda l)/\epsilon_0\ ;\ \vec{\mathbf{E}} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$$

The direction is radial from the cylindrical symmetry.
(The exact formula doesn't matter, only the 1/s dependence does)
Inside the metal cylinder, $\vec{\mathbf{E}} = 0$
Outside the cylinder, the field is as if there were no cylinder (Gauss's law again)



(4 points)

MP: Some learners correctly described all three electric fields but forgot to make a graph. I recommend awarding 1 point for each region.

Typical mistakes:

- Some students approached the problem thinking the space is filled with a dielectric.
- Some students forgot that the inner surface of the conductor has to be negatively charged.
- Graphs missing axes.
- Using a spherical surface instead of a cylindrical one.
- (MP) Some students indicated that a figure would be useful. I didn't add the figure *deliberately* as the learners need to learn to translate the formulation in words into a model

Problem 4 (12 points)

A point charge q is imbedded at the center of a sphere of linear dielectric material with permittivity ϵ and radius R.

A. Calculate the polarization $\vec{\mathbf{P}}$ inside the dielectric to show that

$$\vec{\mathbf{P}} = \frac{q}{4\pi r^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\hat{\mathbf{r}}$$

B. Calculate the surface bound charge density σ_b. Where is the surface bound charge density located?

C. Calculate the volume bound charge density ρ_b. Where is the volume bound charge density located?

Tip: you might find useful the following relation (discussed earlier in the course)

$$\vec{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\vec{\mathbf{r}})$$

D. How are the total charges of the surface bound charge (i.e. integrated over all surface) and the volume bond charge (i.e. integrated over the whole volume), related?

(you don't need to do B and C to answer this question)

Answers:

(problem 4.35 and similar to Example 4.5 solved at lectures)

A. $\oint \vec{\mathbf{D}}\cdot d\vec{\mathbf{a}} = Q_{f_{enc}}$; $\vec{\mathbf{D}} = \frac{q}{4\pi r^2}\hat{\mathbf{r}}$; $\vec{\mathbf{E}} = \frac{1}{\epsilon}\vec{\mathbf{D}} = \frac{q}{4\pi\epsilon r^2}\hat{\mathbf{r}}$; (2 points)

$\vec{\mathbf{P}} = \epsilon_0\chi_e\vec{\mathbf{E}} = \epsilon_0\chi_e\vec{\mathbf{E}} = \epsilon_0\left(\frac{\epsilon}{\epsilon_0} - 1\right)\frac{q}{4\pi\epsilon r^2}\hat{\mathbf{r}} = \frac{q}{4\pi r^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\hat{\mathbf{r}}$ (2 points)

The points are awarded for the correct calculation but not for reproducing the formula

B. $\sigma_b = \vec{\mathbf{P}}\cdot\hat{\mathbf{r}} = \frac{q}{4\pi R^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right)$ (2 points)

It's located at the outer surface of the dielectric (1 point)

C. $\rho_b = -\vec{\nabla} \cdot \vec{\mathbf{P}} = -\frac{q}{4\pi}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\vec{\nabla} \cdot \frac{\hat{\mathbf{r}}}{r^2} = -q\left(1 - \frac{\epsilon_0}{\epsilon}\right)\delta^3(\vec{\mathbf{r}})$ (2 points)
The compensating negative charge is at the center (1 point)

D. They should be equal (2 points)
Quick check:

$$\oint_S \sigma_b \, da = \frac{q}{4\pi R^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right) 4\pi R^2 = q\left(1 - \frac{\epsilon_0}{\epsilon}\right)$$

$$\int_{\mathcal{V}} \rho_b d\tau = -\int_{\mathcal{V}} q\left(1 - \frac{\epsilon_0}{\epsilon}\right)\delta^3(\vec{\mathbf{r}}) d\tau = -q\left(1 - \frac{\epsilon_0}{\epsilon}\right) \text{ OK!}$$

MP: If a learner correctly calculated the total charges but failed to state explicitly that they are equal, I recommend awarding all points.

Note: some learners understood a "sphere" as a shell. In this case, they wouldn't have been able to recover the polarization given in (A) which should have risen concerns to be clarified by teachers. Nevertheless, if the problem is solved correctly and self-consistently assuming the dielectric *shell*, it should be graded for all points.

Typical mistakes:

b): some students forgot to switch the r coordinate for its value R at the surface.
Not as common but still a considerable amount of students used a unit vector in the expression for the surface charge density.
c): some students claimed the divergence to be 0 everywhere or thought the delta function to be 1 at the center (even in spite of direct Tip)