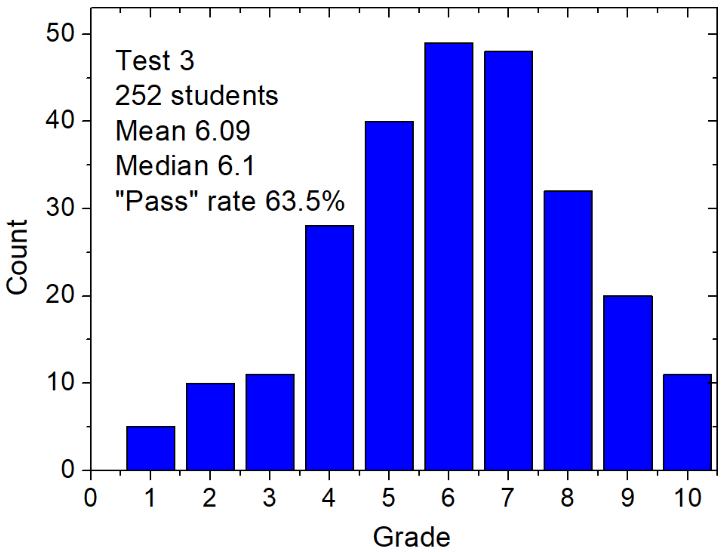
Electricity and Magnetism, Test 3 April 8 2024 18:15-20:15

4 problems, 43 points + 2 bonus points

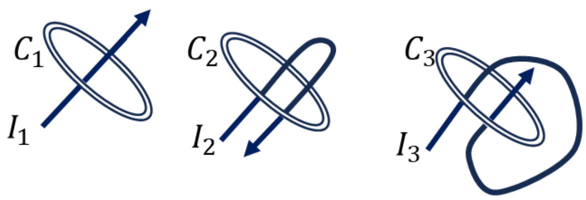
Write your name and student number on **each answer sheet**. Use of a calculator is allowed. You may make use of one A4 (double sided) with handwritten notes and of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\vec{\mathbf{A}}$ is a vector, $\hat{\mathbf{x}}$ is the unit vector into the x-direction. **In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol**.



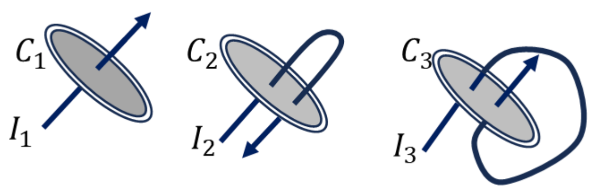
Problem 1. Conceptual questions (12 points; 2 points for each)

In all cases, explain your answers!

A. In Ampère's law formulation, we are talking about the currents (the right-hand side of the equation), “enclosed” into the integration loop (the left-hand side). Which of the following currents $I_1 - I_3$, enclosed by paths C_1, C_2 and C_3, respectively, result in a non-zero enclosed current?



Answers: we imagine a “membrane” with the boundary at the integration path. The currents must pierce the membrane, and their directions also count. I_2 pierces the membrane in opposite direction, hence $I_2^{encl} = 0$.



So the answer is: $I_1$and I_3

B. Could the field given by the following expression, be a magnetic field? (a and b are constants): $\vec{\mathbf{B}}(x, y) = a\cos(bx)\,\hat{\mathbf{x}} + aby\sin(bx)\,\hat{\mathbf{y}}$

Answer: let's calculate $\vec{\mathbf{\nabla}} \cdot \vec{\mathbf{B}}(x, y) = \frac{\partial}{\partial x}(a\cos(bx)) + \frac{\partial}{\partial y}(aby\sin(bx)) = -ab\sin(bx) + ab\sin(bx) = 0$ as it should be for a magnetic field.

So the answer: yes, it could.

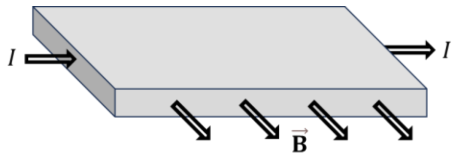
Typical mistakes:

Using the dipole expansion to reason why the field is possible

Not knowing how to calculate the divergence of a vector field

Note by MP: much to my surprise, this appears to be quite difficult for the students – only 40% did it right. $\vec{\mathbf{\nabla}} \cdot \vec{\mathbf{B}}$ is *always* zero (no magnetic monopoles)!

C. A current I flows to the right through a rectangular bar of conducting material, in the presence of a uniform magnetic field $\vec{\mathbf{B}}$ (see the figure). If the moving charges are positive, in which direction are they deflected by the magnetic field?



Answer: (Problem 5.41) the Lorentz force $\vec{\mathbf{F}}_{mag} = Q(\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ and the positive charges flow to the right (with the current direction): they will be deflected down

D. Which of the following operations on the vector potential $\vec{\mathbf{A}}$ will not change the magnetic field:

- Adding a vector constant to $\vec{\mathbf{A}}$
- Adding a divergence of a vector function to $\vec{\mathbf{A}}$
- Adding a gradient of a scalar function to $\vec{\mathbf{A}}$

Answers: a divergence is a scalar so adding it to a vector does not make sense (regardless the vector potential). The other two operations are ok:

Adding a vector constant will not change the magnetic field because of a differential operator (curl)

$$\vec{\mathbf{B}}' = \vec{\mathbf{\nabla}} \times \vec{\mathbf{A}}' = \vec{\mathbf{\nabla}} \times \vec{\mathbf{A}} + \vec{\mathbf{\nabla}} \times \vec{\mathbf{\nabla}}\lambda = \left[rule\ \#10{:}\ \vec{\mathbf{\nabla}} \times \vec{\mathbf{\nabla}}\lambda = 0\right] = \vec{\mathbf{\nabla}} \times \vec{\mathbf{A}} = \vec{\mathbf{B}}$$

Typical mistakes:

Not realizing that divergence of a vector field is a scalar function and can't be added to a vector field (or one can't calculate curl of a scalar function)

Not realizing that gradient of a scalar function is curl free

Note by MP: another problematic question with 43% success rate even though we discussed all but the second option in the lecture.

E. Which of the materials from the table below are attracted to a magnet pole where the magnetic field is non-uniform?

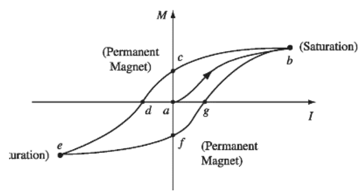
Material	χ_m
Gadolinium	5×10^{-1}
Sodium	8.5×10^{-6}
Copper	-9.7×10^{-6}
Bismuth	-1.7×10^{-4}

Answers: paramagnetics are attracted into a stronger magnetic field. For them, $\chi_m > 0$ so gadolinium and sodium (the other two are diamagnetics)

F. You have a permanent magnet which you would like to de-magnetized (i.e. convert it to a non-magnet). How do you do it?

Answers: there are two ways (either is good as an answer)

1. Apply an external magnetic field that creates magnetization with a direction opposite to that of the magnet. Then increase a magnetic field until the magnet is de-magnetized (points *d* or *g* in the hysteresis loop)

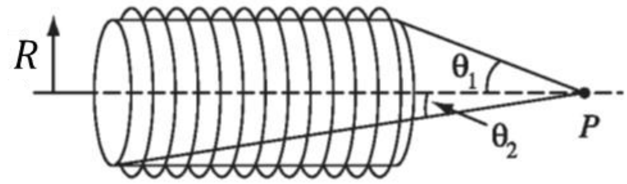


2. Heat it up to the Curie point – the magnetization will be lost due to a phase transition to a paramagnet (= no permanent magnetization)

Problem 2 Bio-Savart law (15 points + 2 bonus points)

(example 5.6 and problem 5.11)

We will calculate the magnetic field on the axis of a tightly wound, *finite-length* solenoid consisting of *n* turns per unit length wrapped around a cylindrical tube of radius *R* and carrying current *I* (see the figure).



A. First, calculate the magnetic field a distance z from the center of a circular loop of radius R, which carries a steady current I, and show that

$$B_z(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$ (3 points)

B. Now find the magnetic field on the axis of the solenoid. Consider the turns to be essentially circular, and use the previous result. Express your answer in terms of θ_1 and θ_2 (5 points)

<u>Tip</u>: you may (or may not) find the following integral useful

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{1}{a^2} \frac{x}{\sqrt{a^2 + x^2}} + const$$

C. Sketch a plot of the magnitude of the magnetic field on the axis of the solenoid as a function of z. Indicate in your plot what power-law behavior results at large positive z, and where the ends of the solenoid are. (You can do this question without succeeding in B) (5 points)

D. What is the field on the axis of an *infinite* (in both directions) solenoid? (2 points)

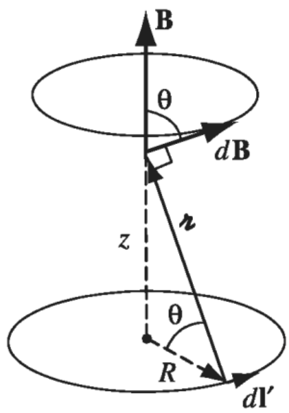
E. Bonus for those who calculated (B): show that your formula gives the right behavior at large distances (2 bonus points)

Tip: for small $\theta, \cos\theta \approx 1 - \dfrac{\theta^2}{2}$

Answers:

A. Example 5.6 (3 points)

The field $d\vec{\mathbf{B}}$ of the segment $d\vec{\mathbf{l}}'$ is directed as shown.



$$\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{I}} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dl'$$

After integration around the current loop, the horizontal component of $\vec{\mathbf{B}}$ cancels out so we're left with the vertical component only (hence, $\cos\theta$ for z-projection of $\vec{\mathbf{B}}$). $\hat{\boldsymbol{\mathscr{r}}}$ and $d\vec{\mathbf{l}}'$ are orthogonal so

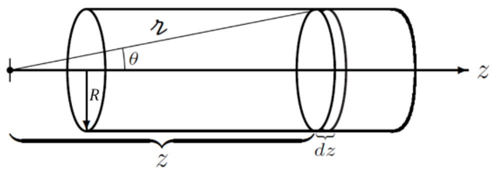
$$B_z(z) = \frac{\mu_0 I}{4\pi}\int \frac{dl'}{\mathscr{r}^2}\cos\theta = \frac{\mu_0 I}{4\pi}\frac{\cos\theta}{\mathscr{r}^2}\int dl' = \frac{\mu_0 I}{4\pi}\frac{R^2}{(R^2+z^2)^{3/2}}2\pi$$

$$= \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

(no points for a simple copy/paste from the formulation)

B. Problem 5.11 (5 points)

We substitute $I \to nI\, dz$ and integrate over dz



$$B = \frac{\mu_0 nI}{2}\int \frac{R^2}{(R^2+z^2)^{3/2}} dz$$

$$R = \sqrt{R^2+z^2}\sin\theta\,; \frac{1}{(R^2+z^2)^{3/2}} = \frac{\sin^3\theta}{R^3}$$

$$z = R\cot\theta\,; dz = -\frac{R}{sin^2\theta}d\theta$$

$$B = \frac{\mu_0 nI}{2}\int R^2\frac{\sin^3\theta}{R^3}\left(-\frac{R}{sin^2\theta}\right)d\theta = -\frac{\mu_0 nI}{2}\int \sin\theta\, d\theta = \frac{\mu_0 nI}{2}\cos\theta|_{\theta_1}^{\theta_2}$$

$$= \frac{\mu_0 nI}{2}(\cos\theta_2 - \cos\theta_1)$$

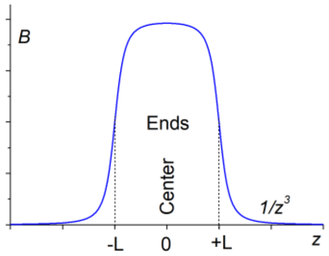
<u>An alternative way</u>: directly integrate over the solenoid length $2L$ (careful with the integration limits!)

$$B = \frac{\mu_0 nI}{2}\int_{z-L}^{z+L}\frac{R^2}{(R^2+z^2)^{3/2}}dz = \frac{\mu_0 nI}{2}\frac{z}{\sqrt{R^2+z^2}}\Bigg|_{z-L}^{z+L}$$

$$= \frac{\mu_0 n I}{2}\left(\frac{z+L}{\sqrt{R^2+(z+L)^2}} - \frac{z-L}{\sqrt{R^2+(z-L)^2}}\right) = \frac{\mu_0 n I}{2}(\cos\theta_2 - \cos\theta_1)$$

Quick check: for an infinitely long solenoid, $\theta_2 = 0, \theta_1 = \pi$, so $\cos\theta_2 - \cos\theta_1 = 1 - (-1) = 2$ and $B = \mu_0 n I$

C. The field inside is constant; the field at long distances scales as z^{-3} (the first non-zero term is a magnetic dipole)



5 points if everything is correct;

-2 points for not showing the functional behavior of the magnetic field at large distances

-1 point if it's shown incorrectly

D. $B = \mu_0 n I$ (2 points)

No derivation is needed; the students should know it

E. Let's check for large distances (=small angles) (L is the solenoid half-length)

$$\cos\theta_2 - \cos\theta_1 \approx 1 - \frac{\theta_2^2}{2} - 1 + \frac{\theta_1^2}{2} \approx \frac{1}{2}\left[\frac{R^2}{(z+L)^2} - \frac{R^2}{(z-L)^2}\right]$$

$$= \frac{R^2}{2}\frac{z^2 - 2zL + L^2 - z^2 - 2zL - L^2}{(z+L)^2(z-L)^2} = -2R^2L\frac{z}{(z^2-L^2)^2} \approx -\frac{2R^2L}{z^3}$$

The power dependence is z^{-3}

2 **bonus** points

Typical mistakes:

2b: Not realizing that the current I should be changed to I*n

2b: Wrong or incomplete substitution of distances and its ratios with trigonometric expressions.

2c: incorrect shape of the field near the edges - it should decrease smoothly, not be constant over the whole solenoid and then start to decrease sharply

2c: Not realizing that the field goes as z^{-3} at large distances, because it can be approximated as a magnetic dipole.

<u>Note by MP</u>: 2a was done correctly by 2/3 of the students but still 1/3 didn't go it even though it was treated at tutorials and was given in the previous year test (as a separate exercise).

2b was designed to be challenging so that the fact that 1/3 of the students did it fully right, is very encouraging!

2e was done correctly by only 2 students (my compliments!), and 9 more attempted to solve it. Of course, it's a bonus question but I'd like to see the challenge taken by more students!

Problem 3. Ampère's law (8 points)

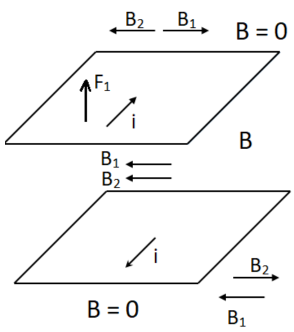
Two infinitely-large parallel surfaces curry uniform non-zero surface currents. Between the plates, the magnetic field is $\vec{\mathbf{B}}$; outside the plates, the magnetic field is zero.

A. Argue that the surface currents are of the same magnitude and counter-propagating (3 points)

B. Find the magnetic force per unit area on one of the plates, expressed only in the magnetic field B (not K). Don't forget to indicate a direction in your final answer. (5 points)

Tip: first find K in terms of B

Answers: Example 5.7 and Problem 5.17 from tutorials modified



A. Let the surface current run into the x-direction: $\vec{\mathbf{K}} = K\,\hat{\mathbf{x}}$. Then the magnetic field of one plate is homogeneous and equal to

$$\vec{\mathbf{B}} = \begin{cases} +(\mu_0 K/2)\,\hat{\mathbf{y}} \text{ for } z < 0 \\ -(\mu_0 K/2)\,\hat{\mathbf{y}} \text{ for } z > 0 \end{cases}$$

To nullify the field outside the planes, the currents should run in the opposite directions and be of the same magnitude.

3 points; -1 point for each incorrect argument

Students don't need to derive $\vec{\mathbf{B}}$ if they remember it

B. The magnitude of the magnetic field between the plates $B = \mu_0 K$ so

$$K = \frac{B}{\mu_0} \text{ (2 points)}$$

$$\vec{\mathbf{F}}_{mag} = \int (\vec{\mathbf{K}} \times \vec{\mathbf{B}})\, da$$

$$\vec{\mathbf{f}} = \frac{\vec{\mathbf{F}}_{mag}}{da} = (\vec{\mathbf{K}} \times \vec{\mathbf{B}}) = K\,\hat{\mathbf{x}} \times (\mu_0 K/2)\,\hat{\mathbf{y}} = \frac{\mu_0 K^2}{2}\,\hat{\mathbf{z}}$$

$$= \frac{\mu_0 (B/\mu_0)^2}{2}\,\hat{\mathbf{z}} = \frac{B^2}{2\mu_0}\,\hat{\mathbf{z}}$$

The force will repel the plates (as it should be for the oppositely-directed currents)

3 points; -1 point for not indicating / indicating wrongly the direction

Typical mistakes:

3a: Simply restating what was to be shown without much elaboration while the question specifically asked for "arguing"

3b: Forgetting the formula for force in terms of K and using magnetization instead (why magnetization?)

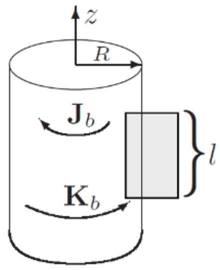
3b: Using a dot product instead of a cross product

Problem 4. **Magnetization and bound currents (8 points)**

An infinitely long cylinder, of radius R, carries a "frozen-in" magnetization, parallel to the axis, $\vec{\mathbf{M}} = ks\hat{\mathbf{z}}$, where k is a constant and s is the distance from the axis; there is no free current anywhere.

A. Locate all the bound currents, calculate them and draw them schematically in a copy/sketch of the cylinder. Don't forget about their directions! (4 points)

B. Find the magnetic field, produced by the magnetization, inside and outside the cylinder (4 points)



Answers: Problem 6.12; similar was done in the lecture

A. $\vec{\mathbf{J}}_b = \vec{\nabla} \times \vec{\mathbf{M}} = -\frac{\partial}{\partial s}(ks)\hat{\boldsymbol{\varphi}} = -k\hat{\boldsymbol{\varphi}}$; located inside the cylinder

$\vec{\mathbf{K}}_b = \vec{\mathbf{M}} \times \hat{\mathbf{n}} = kR\hat{\mathbf{z}} \times \hat{\mathbf{n}} = kR\hat{\boldsymbol{\varphi}}$;

located at the surface of the cylinder
See the figure for their directions

4 points; -1 point for each incorrect calculation and/or wrong direction

B. Method 1

By symmetry, $\vec{\mathbf{H}}$ points into the $\hat{\mathbf{z}}$ direction. Let's take an amperian loop as shown in the figure.

$\oint \vec{\mathbf{H}} \cdot d\vec{l} = I_{f\,enc} = 0$ since there is no free current here $\vec{\mathbf{H}} = 0$; $\vec{\mathbf{B}} = \mu_0\left(\vec{\mathbf{H}} + \vec{\mathbf{M}}\right) = \mu_0 ks\hat{\mathbf{z}}$

Outside the cylinder, $\vec{\mathbf{M}} = 0$ so $\vec{\mathbf{B}} = 0$

Method 2

We notice that the bound currents essentially represent a superposition of solenoids so that $\vec{\mathbf{B}}$ inside is in the $\hat{\mathbf{z}}$ direction while outside $\vec{\mathbf{B}} = 0$

$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I_{enc}; Bl = \mu_0\left[\iint J_b da + K_b l\right] = \mu_0[-kl(R-s) + kRl] = \mu_0 kls$

$\vec{\mathbf{B}} = \mu_0 ks\hat{\mathbf{z}}$

4 points; -1 point for each incorrect direction; -1 point for not giving the field outside

Typical mistakes:

In many answers, the surface bound current was proportional to s, instead of R. All the current should be distributed over the surface where $s = R$.

Some experienced difficulties with choosing a proper Amperian loop that would enclose the appropriate current.