

Electricity and Magnetism Test 4

7 May 2024, 18:30-20:30

The maximum score is 43 points. Good luck!

Solution: Please grade as follows:

- If several students make a similar mistake, please write that down AND agree on a consistent way to score it!
- Only award full integer points equal or larger than zero. No fractional points.
- If the question asks for an explanation, calculation, determination, argumentation, etc., award no points if this is missing or clearly incorrect / incoherent.
- Subtract points for each mistake only once, unless the error substantially simplifies or alters the rest of the problem.
- Pay close attention to answers for ‘show that ...’ questions. Making two mistakes that miraculously cancel each other should be awarded *fewer* points than making one mistake and not reaching the result.

I. Short questions [19 points]

1. (5 points) Consider a battery with internal resistance through which a steady current flows. Make a drawing (no explanation needed) indicating:
 - The directions, rough relative magnitudes, and labels of the effective forces per unit positive moving charge *inside* the battery;
 - the direction in which the (positive) current flows;
 - which end of the battery has ‘+’ printed on it.

Solution: One point each for:

- $\mathbf{f}_\rho$ / resistive force drawn and directed opposite to the current
- $\mathbf{f}_s$ / source force / battery force drawn and directed along the current
- $\mathbf{E}$ / $-\nabla V$ / electric force / electrostatic force drawn and directed opposite to $\mathbf{f}_s$
- Forces are in equilibrium (i.e. $\mathbf{f}_s$ is roughly as large as $-\nabla V + \mathbf{f}_\rho$). Do not award if no forces are drawn at all.
- The current flows from - to + (unlike in a resistor).

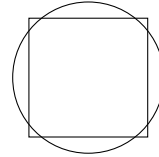
Subtract one point for each other, spurious force drawn.

If someone draws a tiny arrow for gravity AND makes a correct note that gravity does not scale with charge: no penalty. If such a note is missing, subtract one point (the question asks for 'forces *per* unit positive charge').

If someone omits the resistive force, but the diagram clearly implies a net force directed opposite to the current AND there is a correct note about resistive processes happening (e.g. 'occasional jerky forces from collisions'), no penalty.

Common mistake: forgetting the resistive force (and adding no note about it).

2. (3 points) A circular wire loop lies in a uniform external magnetic field that points out of the page. You rearrange the wire into a square with an identical circumference, as drawn accurately in the figure on the right. Explain what the average direction of the current in the loop is during this procedure (as seen from above).



Solution: The surface area decreases, as is clear from the diagram.

(Or a short computation: same circumference means $2\pi r = 4d$ where d is a side of the square; area of square $d^2 = (\pi r/2)^2 = \pi r^2 \cdot (\pi/4)$, less than area of circle (πr^2).)

Thus the magnetic flux change is downwards, so the emf and therefore the current is counter-clockwise. (The right-hand rule associates downwards with clockwise, but there is a minus sign in Faraday's law.)

One point each for:

- The surface area decreases (clear from the diagram / small calculation)
- Correct implication for the change in the magnetic flux
- Correct implication for the direction of (the Faraday emf and therefore) the current

Concluding the area increases, then reasoning correctly from this premise (reaching the opposite answer): maximum of 2 points.

Stating that the area decreases, but using an incorrect calculation to confirm the decrease: no penalty, since we do not penalize having no calculation at all.

Concluding the area stays constant, then reasoning correctly from this premise (answer: no current flows): maximum of 1 point. This is a substantial simplification of the question.

Common mistake: no explanation for the appearance of the current during the transformation of the loop. (Some students assumed the current was already flowing in the square loop?)

3. (2 points) Suppose $\mathbf{E} = y\hat{\mathbf{x}} + xy\hat{\mathbf{y}}$. Determine if the *magnetic* field is time-dependent.

Solution: $-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{\mathbf{z}} = (y-1)\hat{\mathbf{z}}$, so yes, the magnetic field is time-dependent.

One point each for:

- Using Faraday's differential law, $-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$
- Correct calculation and conclusion

4. (3 points) Explain why unplugging a device that draws a significant current is likely to generate sparks.

Solution: The device's inductance resists changes in the current through it. Unplugging suddenly means dI/dt is highly negative, so the inductance of the device gives a high (back) emf in the direction of the the original current. The resulting voltage may be high enough to cause sparks in air.

One point each for:

- Unplugging means a very sudden change in current
- The device's inductance then causes a high (back) emf in the original current's direction
- Which causes a high voltage / dielectric breakdown, explaining the sparks

To award the last point, either some implication on the voltage must be drawn from the emf, or 'dielectric breakdown' must be mentioned. (In principle both are part of an exhaustive explanation.)

Common mistakes were mostly incomplete explanations.

5. (3 points) We put a sinusoidal oscillating (alternating) current through the first coil of a transformer. The second coil will then also show an oscillating current. Determine by what fraction of a period the second coil's sinusoid lags behind the first. Ignore self-inductance.

Solution: Let $I_1 = \sin \omega t$, then

$$I_2 \propto \mathcal{E}_2 = -M \frac{dI}{dt} \propto -\cos \omega t$$

which is 1/4 period delayed (or 3/4 period early).

One point each for

- Using $I \propto \mathcal{E}$ / Ohm's law
- Using $\mathcal{E} = -M \frac{dI}{dt}$
- Correct conclusion. Also allow 3/4 delayed (1/4 early) which might refer to some notion of phase shift (or just because we are nice / we also get this wrong a lot).

6. (3 points) Can displacement currents produce a time-independent magnetic field? Explain using a Maxwell equation.

Solution: The displacement current is $\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$, which produces a curl of $\mathbf{B}$ through [Maxwell-Ampere equation]. The produced $\mathbf{B}$ can only be time-independent if $\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ is.

This is physically not possible forever / is possible as long as $\mathbf{E} \propto t$.

One point each for:

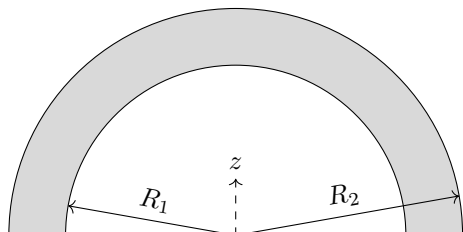
- Realizing displacement current is $\propto \frac{\partial \mathbf{E}}{\partial t}$
- Realizing displacement current relates to the magnetic field via the Maxwell-Ampere law
- Concluding it is possible if/while $E \propto t$, or concluding it is impossible because forever-increasing fields are impossible

Misinterpreting displacement current as ordinary current (and using Ampere's law): no points, substantial simplification.

Common mistake: saying E (rather than its rate of change) must be constant for B to be time-independent.

II. Hemispherical resistor [9 points]

Two concentric metal hemispheres (half-spheres) of radii R_1 and R_2 are connected by ohmic material of resistivity ρ . The inner and outer hemispheres are connected to the $+$ and $-$ of a battery, respectively, and a steady current flows. The figure below shows a cross-section of the setup; the shaded area is the ohmic material.



7. (3 points) Show/argue that the electric field in the ohmic material between the hemispheres is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2} \hat{\mathbf{r}} \quad (1)$$

where q_1 is the total charge on the inner hemisphere, and $\mathbf{r}$ is the distance from the origin of the spheres.

Solution: The given field puts each hemisphere at equipotential (because the field has no angular components) and is parallel to the radial boundary surfaces. It also puts no charge inside the resistor (because the divergence is zero except at $r = 0$, which is not in the material) / is the field of a point charge at the origin. By the electrostatic uniqueness theorem, the field must be as given.

- Recognizing the given field is that of a point charge at the origin / implies no charge (divergence) in the material
- Recognizing the hemispheres would each be at equipotential in the given field
- Recognizing the given field is parallel to the exposed material surfaces

Award no points for arguing as if we are in empty space, e.g. saying the inner hemisphere's field must be the same as that of a point charge. (The inner hemisphere is not a full sphere. If the ohmic material was absent, the field between the hemispheres would be more complicated.)

8. (3 points) Show that the current is radial and

$$I = \frac{q_1}{2\epsilon_0\rho}. \quad (2)$$

Solution: By Ohm's law, $\mathbf{J} = \mathbf{E}/\rho$. The current, like the given field, is radially outward and rotationally symmetric over the hemisphere, thus

$$I = \int \mathbf{J} \cdot d\mathbf{a} = JA = \frac{1}{4\pi\epsilon_0\rho} \frac{q_1}{r^2} (2\pi r^2) = \frac{q_1}{2\epsilon_0\rho}$$

One point each for:

- Correctly using $\mathbf{J} = \mathbf{E}/\rho$
- Correctly using $I = \int \mathbf{J} \cdot d\mathbf{a}$
- Correct conclusion

9. (3 points) Compute the resistance of this setup.

Solution: The potential difference is

$$\Delta V = \int_{R_1}^{R_2} \mathbf{E} \cdot d\mathbf{l} = \frac{q_1}{4\pi\epsilon_0} \left(-\frac{1}{R_2} + \frac{1}{R_1} \right)$$

Thus the resistance is

$$R = \frac{\Delta V}{I} = \frac{\rho}{2\pi} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

One point each for:

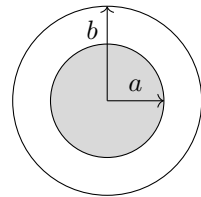
- Correctly using $\Delta V = \int_{R_1}^{R_2} \mathbf{E} \cdot d\mathbf{l}$
- Correctly using $R = \Delta V / I$
- Correct conclusion

Common mistakes (besides integration errors): wrong integration bound order / forgetting $\mathbf{E}$ is the negative gradient of V . I also make this mistake often, but then I usually realize a negative resistance is not great.

III. Coaxial cable [15 points]

Consider a long coaxial cable. A cross-section of the cable is shown in the figure on the right. Let s be the distance from the symmetry axis.

- A steady current I flows homogeneously (out of the paper) through a cylindrical solid wire that spans $s < a$.
- There is vacuum at $a < s < b$.
- At $s = b$, the current returns (into the paper) through a thin coaxial cylindrical shell.



10. (4 points) Argue that the magnetic field is:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \hat{\phi} \begin{cases} s/a^2 & s < a \\ 1/s & a < s < b \\ 0 & s > b \end{cases} \quad (3)$$

(Here $\hat{\phi}$ rotates circumferentially around the symmetry axis, in the direction given by the right-hand rule.)

Solution: The current is axial and homogeneous, so the magnetic field is circumferential ($\propto \hat{\phi}$). Draw a circular Amperian loop of radius s :

$$\mu_0 I_{\text{encl}} = 2\pi s B$$

but I_{encl} is zero outside the outer cylinder, I between the cylinders, and $I(s^2/a^2)$ inside the wire (s^2/a^2 is the ratio of enclosed and total areas, and thus $\propto I_{\text{encl}}/I$). From this the given expression follows.

One point each for:

- Correct argument that $\mathbf{B}$ is circumferential
- Correct use of Ampere's law
- Correct I_{encl}
- Correct final answer

11. (4 points) Calculate the self-inductance of a length l of this cable.

Solution: The magnetic energy density is

$$\frac{B^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2} \begin{cases} s^2/a^4 & a < s \\ 1/s^2 & a < s < b \\ 0 & b < s \end{cases}$$

Integrating in cylindrical coordinates: $\int \dots d\tau = \int \dots s ds d\phi dz = 2\pi l \int \dots s ds$:

$$\begin{aligned} W &= \frac{\mu_0 I^2 l}{4\pi} \left(\int_0^a \frac{s^3}{a^4} ds + \int_a^b \frac{1}{s} ds \right) \\ &= \frac{\mu_0 I^2 l}{4\pi} \left(\int_0^a \left[\frac{s^4}{4a^4} \right]_0^a + \left[\ln s \right]_a^b \right) \\ &= \frac{\mu_0 I^2 l}{4\pi} \left(\frac{1}{4} + \ln \frac{b}{a} \right) \end{aligned}$$

From $W = \frac{1}{2} LI^2$, $L = 2W/I^2$, we get the desired result:

$$L = \frac{\mu_0 l}{2\pi} \left(\frac{1}{4} + \ln \frac{b}{a} \right)$$

One point each for:

- Insight that energy density is $\propto B^2$
- Integrating this over volume with $d\tau = s ds d\phi dz$
- Using $W = \frac{1}{2} LI^2$
- Correct final answer

Common mistakes: forgetting s in the Jacobian; using $\Phi = LI$, which applies to line currents, then getting an interesting but wrong answer. (As noted in the mock exam, you can sometimes Frankenstein this to work for more general currents, but it is perilous. Don't do it.)

For the rest of the problem, consider the time-dependent (alternating) current:

$$I(t) = I_0 \cos \omega t. \quad (4)$$

Here t indicates time and I_0 and ω are positive constants. Assume $\mathbf{B}$ is exactly given by equation 3, with the I of equation 4.

12. (2 points) Explain, using a Maxwell equation, why this assumption about $\mathbf{B}$ is reasonable only if ω is small.

Solution: A large ω would rapidly vary the electric fields (both the induced field caused by the changing magnetic field, and the ohmic field inside the conductors). This implies large displacement currents that affect the magnetic field.

One point each for

- Large fluctuating currents are associated with fluctuating electric fields
- These would affect change the magnetic field through the Maxwell-Ampere equation

Alternative: The magnetic field is calculated based only on Faraday's law / using Faraday quasistatics. This theory is only valid when changes with time are small, which requires a small ω .

One point each for

- The magnetic field was calculated using Faraday's law alone / Faraday quasistatics
- An assumption of this theory is that changes with time are slow

Interesting mistake: saying B gets a divergence at high frequency (please write a paper if you see this!)

13. (5 points) Calculate the induced/Faraday electric field in the region where $s < a$ (inside the inner wire), including its direction relative to the current. Ignore other contributions to the electric field.

Solution: See an example solution below.

One point each for:

- Realizing $\mathbf{E}$ is axial ($\propto \hat{\mathbf{z}}$)
- Correctly using $\Phi = \int \mathbf{B} \cdot d\mathbf{a}$
- Correctly using Faraday's integral law
- Correct conclusion for the magnitude, up to an s -independent constant. For example, allow either

$$E_z = \frac{\mu_0 I_0 \omega \sin(\omega t)}{2\pi} \left(\frac{1}{2} - \frac{s^2}{2a^2} + \ln \frac{b}{a} \right)$$

(the correct answer), or

$$E_z = -\frac{\mu_0 I_0 \omega \sin(\omega t) s^2}{4\pi a^2} + K$$

(for taking both sides of the loop inside the wire), or

$$E_z = -\frac{\mu_0 I_0 \omega \sin(\omega t) s^2}{4\pi a^2}$$

(for assuming $E = 0$ at the center). Also award this point, but not the next one, if there is a minus sign error in these.

- Correct direction / opposite to the CHANGE in the current. (For example, for the $E \rightarrow 0$ at ∞ answer, parallel to the current just after $t = 0$, opposite to it just before.)

Doing $\oint \mathbf{E} \cdot d\ell = 2El$ is incorrect. Even if E were constant in s (which it is not), you'd get a zero $\oint \mathbf{E} \cdot d\ell$; one side adds, another subtracts. Do not award the third point. But award the other points if the rest of the calculation is correct (the result will be off by a factor 2).

Using $\nabla \times \mathbf{B}$ to compute $\frac{\partial E}{\partial t}$, then integrating over t : this makes no sense, as it assumes all of $\nabla \times \mathbf{B}$ is set by the displacement current. By assuming B is fixed to the value caused by the conduction current, we assume the displacement current is negligible. (The discussion on small ω in the last question also helps you realize this.) Do not award points 2-4; the first and final could still be awarded for arguments independent of the incorrect calculation.

We decided to allow any field up to a space-independent constant because constant fields in quasistatics for infinite current distributions are a subtle issue. In this problem, we have equal currents going in opposite directions, which is a good argument for assuming that any field must quickly vanish outside the cable. In other problems this is less clear, e.g. example 7.9 in Griffiths. Moreover, there is a technical ambiguity defining what an ‘induced’ vs. a ‘Coulomb’ electric field means up to a constant. For more discussion, see e.g. <https://pubs.aip.org/aapt/ajp/article-abstract/54/10/946/1045931> and <https://pubs.aip.org/aapt/ajp/article/54/12/1142/1041499>.

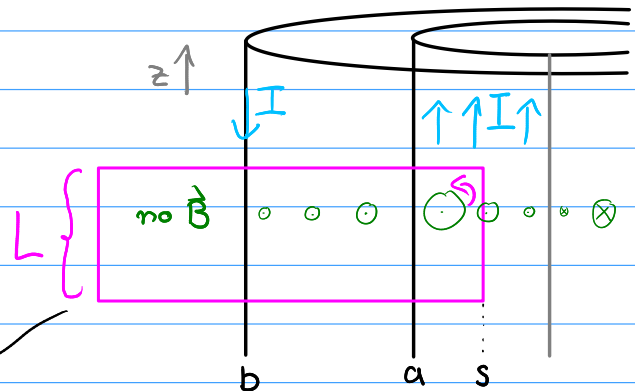
Common mistakes: using a circumferential loop (which encloses no flux change, the field is circumferential), or using incorrect integration bounds.

- $\frac{\partial B}{\partial t}$ is $\propto \hat{\phi} \Rightarrow E \propto \hat{z}$.

- No flux change through here $\hookrightarrow$

$\Rightarrow E$ independent of s
for $s > b$.

Only $E=0$ makes sense.



- $\oint \vec{E} \cdot d\vec{\ell} = - \frac{d\Phi}{dt}$ (Faraday)

$$E_z(s) \cdot L = - \int \frac{\partial B}{\partial t} \cdot d\vec{a} = -L \int_s^b \frac{\partial B(s')}{\partial t} ds'$$

$$= - \frac{\mu_0 L \dot{I}}{2\pi} \left(\int_s^a \frac{s'}{a^2} ds' + \int_a^b \frac{1}{s'} ds' \right) \quad (\text{for } s < a)$$

$$= \quad \left(\left[\frac{s'^2}{2a^2} \right]_s^a + \left[\ln s' \right]_a^b \right)$$

$$= \quad \left(\frac{1}{2} - \frac{s^2}{2a^2} + \ln \frac{b}{a} \right)$$

$\hookrightarrow \dot{I} = -I_0 \sin \omega t$

$$\Rightarrow E(s) = \frac{\mu_0 I_0 \omega \sin(\omega t)}{2\pi} \underbrace{\left(\frac{1}{2} - \frac{s^2}{2a^2} + \ln \frac{b}{a} \right)}_{>1} \quad \text{for } s < a$$

$\Rightarrow$ induced E opposite to $\dot{I}$,
as expected from Lenz.