Electricity and Magnetism Test 5

24 May 2024, 18:30-20:30

Use your double-sided A4 cheat sheet and the provided formula sheet. A calculator is not needed.

- Clearly sketch or describe paths, surfaces, and volumes you integrate over.
- If you use that X is zero because of law Y, *say* X is zero because of law Y.

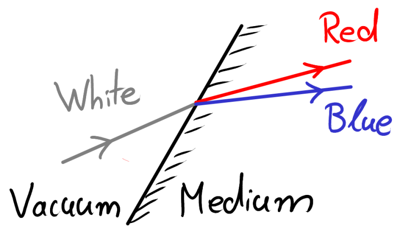
Good luck!

I. Short questions [15 points]

1. (4 points) A traveling monochromatic electromagnetic plane wave in vacuum is given by the real part of
$$\tilde{\mathbf{E}} = E_0\, e^{7i}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{x}}. \tag{1}$$
with $\mathbf{k} = (0, 0, -k)$. Write down the real, physical, **magnetic** field associated with this wave. Do not introduce new symbols without clearly defining them.

2. (4 points) Derive the following wave equation for $\mathbf{E}$ in a linear ohmic medium with conductivity σ, permittivity ϵ, permeability μ, and no free charge:
$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial \mathbf{E}}{\partial t}. \tag{2}$$
You may use Maxwell's equations in generic *linear* media; you do not have to derive these if you remember them.

3. (4 points) The figure on the right shows a beam of white light dispersing at an interface from vacuum into a non-magnetic linear medium ($\mu = \mu_0$). Using the figure, explain whether the permittivity increases or decreases with frequency in this medium.



4. (3 points) Explain which polarization direction (horizontal, vertical, some angle with respect to the road ...) sunglasses should *block* to reduce the glare of a wet road viewed at an angle close to Brewster's angle for the air-water interface.

II. Energy conservation [8 points]

Consider a long, tightly wound solenoid with n windings per unit length, radius R, and a slowly changing current $I(t)$. Let s be the radial distance from the symmetry axis. Recall that the quasistatic magnetic field is
$$\mathbf{B} = \begin{cases} \mu_0 I(t) n\hat{\mathbf{z}} & s < R \\ 0 & s > R \end{cases}. \tag{3}$$

5. (3 points) Show, within the Faraday quasistatics approximation and assuming $\mathbf{E} = 0$ at $s = 0$, that
$$\mathbf{E} = -\frac{\mu_0 n}{2}\dot{I}(t)\hat{\boldsymbol{\phi}}\begin{cases} s & s < R \\ R^2/s & s > R \end{cases}. \tag{4}$$

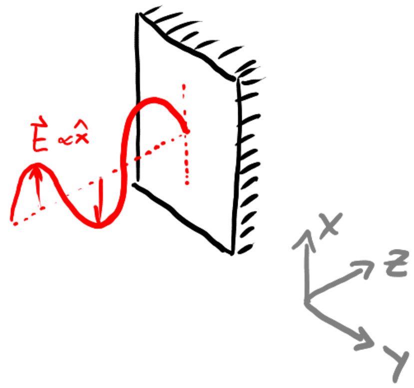
As usual, $\dot{I}(t) = dI/dt|_t$.

6. (2 points) Assuming this electric field and the quasistatic magnetic field, find the Poynting vector in the interior of the solenoid.

7. (3 points) Show/argue that the fields given above violate local energy conservation if $\ddot{I}(t) \neq 0$. Your choice which region you consider.

III. Mirror under pressure [15 points]

A perfect uncharged conductor with $\mu = \mu_0$ and $\epsilon = \epsilon_0$ occupies the half of space with $z > 0$. It allows neither electric nor magnetic fields in its interior. The rest of space is vacuum.



Consider the magnetic field just on the $z < 0$ side of $z = 0$. We will call this $\mathbf{B}(z = 0^-)$. The x and y axes are parallel to the interface as shown.

8. (4 points) By integrating a Maxwell equation, prove that

$$B_{\mathrm{y}}(z = 0^-) = \mu_0 K_x. \tag{5}$$

Here $\mathbf{K}$ is the surface current on the conductor surface.

A monochromatic electromagnetic plane wave polarized in the x direction propagates along z. It hits the conductor and is perfectly reflected, producing a similar wave propagating along $-z$. Thus, on the vacuum side in our situation,

$$\mathbf{E}_I = E_0 \cos(kz - \omega t)\widehat{\mathbf{x}} \tag{6}$$

$$\mathbf{E}_R = E_0 \cos(-kz - \omega t + \phi)\widehat{\mathbf{x}}. \tag{7}$$

9. (3 points) Explain using a boundary condition that the phase shift ϕ must be π in this case. State exactly what boundary condition you are using **AND** which Maxwell's equation it is derived from (you do not have to do the derivation).

   Hint: what is the consequence of a phase shift of π?

10. (3 points) Show that

$$\mathbf{B}(z{=}0^-) = \frac{2E_0}{c}\cos(\omega t)\widehat{\mathbf{y}}. \tag{8}$$

On the $z > 0$ of the interface, the magnetic field is zero. (The surface current explains the discontinuity.) The average magnetic field

$$\mathbf{B}_{\mathrm{avg}} = \frac{\mathbf{B}(z = 0^-) + \mathbf{B}(z = 0^+)}{2} = \frac{\mathbf{B}(z = 0^-)}{2} \tag{9}$$

acts on the surface current to give a Lorentz force.

11. (5 points) Show explicitly that the time-averaged Lorentz force on the surface current equals twice the time-averaged radiation pressure in the incident wave (as it should, by conservation of momentum).

When you are finished: Write your name and student number on your solutions – both sheets if you used two. Place solutions in the box with your student number. Add a mark next to your name on the list in that box. Return the formula sheet and unused paper to their corresponding stacks. Take the piece of paper you are currently reading home. Exit the hall in the back, not where you entered.