Electricity and Magnetism Test 5

24 May 2024, 18:30-20:30

Use your double-sided A4 cheat sheet and the provided formula sheet. A calculator is not needed.

- Clearly sketch or describe paths, surfaces, and volumes you integrate over.
- If you use that X is zero because of law Y, *say* X is zero because of law Y.

Good luck!

Solution: Please grade as follows:

- If several students make a similar mistake, please write that down AND agree on a consistent way to score it!
- Only award full integer points equal or larger than zero. No fractional points.
- If the question asks for an explanation, calculation, determination, argumentation, etc., award no points if this is missing or clearly incorrect / incoherent.
- Subtract points for each mistake only once, unless the error substantially simplifies or alters the rest of the problem.
- Pay close attention to answers for 'show that ...' questions. Making two mistakes that miraculously cancel each other should be awarded *fewer* points than making one mistake and not reaching the result.

I. Short questions [15 points]

1. (4 points) A traveling monochromatic electromagnetic plane wave in vacuum is given by the real part of

$$\tilde{\mathbf{E}} = E_0\, e^{7i}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{x}}. \tag{1}$$

with $\mathbf{k} = (0, 0, -k)$. Write down the real, physical, **magnetic** field associated with this wave. Do not introduce new symbols without clearly defining them.

Solution: One of these (or equivalent forms) is correct

$$\mathbf{B} = -\frac{E_0}{c}\cos(-kz - \omega t + 7)\hat{\mathbf{y}}$$
$$\mathbf{B} = -\frac{E_0}{c}\cos(kz + \omega t - 7)\hat{\mathbf{y}}$$
$$\mathbf{B} = -\frac{E_0}{c}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + 7)\hat{\mathbf{y}}$$
$$\mathbf{B} = -\frac{E_0}{c}\cos(-\mathbf{k}\cdot\mathbf{r} + \omega t - 7)\hat{\mathbf{y}}$$

One point each for

- Magnitude E_0/c
- Correct phase and time dependence $(-\omega t + 7)$, just like $\mathbf{E}$

- Correct polarization (from $\tilde{\mathbf{B}} = \frac{\hat{\mathbf{k}}\times\tilde{\mathbf{E}}}{c}$)
- Correct conclusion / use of Euler's formula to get the real part

Many students instead gave

$$\mathbf{B} = -\frac{E_0}{c}\cos(7)\cos(-kz-\omega t)\widehat{\mathbf{y}}$$

hoping that $\Re(ab) = \Re(a)\Re(b)$ (where $\Re$ means 'take real part of'). This is false: the real part of $2i$ is zero, not two. Maximum 3 points.

Of course another common mistake was forgetting the minus sign in the cross product.

Some students gave the real part of the electric field (despite the bold instruction), so e.g.

$$\mathbf{E} = E_0\cos(-kz-\omega t+7)\widehat{\mathbf{x}}$$

This is worth a maximum 2 points (the phase and Euler are done correctly).

2. (4 points) Derive the following wave equation for $\mathbf{E}$ in a linear ohmic medium with conductivity σ, permittivity ϵ, permeability μ, and no free charge:

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}. \tag{2}$$

You may use Maxwell's equations in generic *linear* media; you do not have to derive these if you remember them.

Solution: Take the curl of Faraday's law, then use the Maxwell-Ampere law:

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{E}) &= -\frac{\partial}{\partial t}\nabla\times\mathbf{B}\\ &= -\frac{\partial}{\partial t}\left(\mu\mathbf{J}_{\text{free}} + \mu\epsilon\frac{\partial\mathbf{E}}{\partial t}\right) \quad \text{[1pt]}\\ &= -\mu\sigma\frac{\partial\mathbf{E}}{\partial t} - \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} \quad \text{[1pt]}\end{aligned}$$

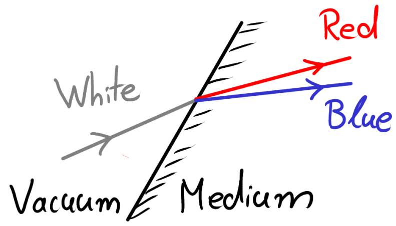
where in the last line we used Ohm's law($\mathbf{J}_{\text{free}} = \sigma\mathbf{E}$). For the left-hand side we use second-derivative identity number 11:

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} \quad \text{[1pt]}.$$

But by Gauss's law, $\nabla\cdot\mathbf{E} = \rho_{\text{free}} = 0$ [1pt]. The result follows.

Be lenient towards a small notation mistake (spurious 0 subscripts on μ or ϵ, omitting 'free' subscripts on charge, no vector arrows, etc.). If there are several, subtract a point.

3. (4 points) The figure on the right shows a beam of white light dispersing at an interface from vacuum into a non-magnetic linear medium ($\mu = \mu_0$). Using the figure, explain whether the permittivity increases or decreases with frequency in this medium.



Solution:

- Blue light has the higher ω [1pt].
- From the figure, it shows the most refraction [1pt].
- Thus (by Snell's law) it has the highest $n \propto \sqrt{\epsilon/\epsilon_0}$ and therefore ϵ [1pt].
- So ϵ increases with frequency in this medium. [1pt]

Some students did not use the figure, and instead argued that dipoles in the medium have a harder time oscillating with higher frequencies. This instead leads to ϵ decreasing with ω, and is indeed seen for water at microwave and radio frequencies. In the visible-light region, water and other transparent media show the opposite behavior. And the question clearly says to use the figure…

If you thought red has higher frequency, but reasoned consistently from that (concluding ϵ decreases with ω), you lose only one point for this. Remember red is close to infrared, and violet to ultraviolet.

4. (3 points) Explain which polarization direction (horizontal, vertical, some angle with respect to the road …) sunglasses should *block* to reduce the glare of a wet road viewed at an angle close to Brewster's angle for the air-water interface.

Solution:

- At Brewster's angle, light polarized parallel to the plane of incidence (normal to the interface, p-polarization) is not reflected / purely transmitted (and absorbed by the road surface). [1pt]
- The wet road is a horizontal interface, so the plane of incidence is vertical. [1pt]
- Thus (incident under Brewster's angle) the road only emits horizontally polarized light, and the sunglasses should block this. [1pt]

'Vertical polarization because light polarized parallel to the surface is absorbed at Brewster's angle': 1pt. This is close to the meaning of Brewster's angle but mixes up the plane of incidence and the surface (a substantial simplification).

Some students tried calculating Brewster's angle for the interface, rather than answering the question. Note also the exam told you a calculator was not needed…

II. Energy conservation [8 points]

Consider a long, tightly wound solenoid with n windings per unit length, radius R, and a slowly changing current $I(t)$. Let s be the radial distance from the symmetry axis. Recall that the quasistatic magnetic field is

$$\mathbf{B} = \begin{cases} \mu_0 I(t) n \hat{\mathbf{z}} & s < R \\ 0 & s > R \end{cases}. \tag{3}$$

5. (3 points) Show, within the Faraday quasistatics approximation and assuming $\mathbf{E} = 0$ at $s = 0$, that

$$\mathbf{E} = -\frac{\mu_0 n}{2} \dot{I}(t) \hat{\phi} \begin{cases} s & s < R \\ R^2/s & s > R \end{cases}. \tag{4}$$

As usual, $\dot{I}(t) = dI/dt|_t$.

Solution: Since **B** is axial, **E** is circumferential. Use Faraday's integral law

$$\oint \mathbf{E} \cdot d\ell = -\frac{d\Phi}{dt}$$

with a circular loop of radius s. [1pt]

On the left-hand side, we get $2\pi s E_\phi(s)$. [1pt] The flux through the loop is

$$\Phi = B_z \begin{cases} \pi s^2 & s < R \\ \pi R^2 & s > R \end{cases}$$

since B_z is uniform. Thus

$$\mathbf{E} = -\frac{\mu_0 \dot{I}(t) n}{2\pi s} \phi \begin{cases} \pi s^2 & s < R \\ \pi R^2 & s > R \end{cases}$$

from which the result follows. [1pt]

(How can there be an **E** at $s > r$ if $\mathbf{B} = 0$ there? Well, $\nabla \times \mathbf{E} = 0$ there by Faraday's law, but that does not mean $\mathbf{E} = 0$! For example, in magnetostatics, $\nabla \times \mathbf{B} = 0$ outside a current-carrying wire; but it definitely has a magnetic field around it.)

6. (2 points) Assuming this electric field and the quasistatic magnetic field, find the Poynting vector in the interior of the solenoid.

Solution: For $s < R$:

$$\begin{aligned} \mathbf{S} &= \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \quad \text{[1pt]} \\ &= -\frac{\mu_0^2 I(t) \dot{I}(t) s n^2}{2\mu_0} (\hat{\phi} \times \hat{\mathbf{z}}) = -\frac{1}{2} \mu_0 I(t) \dot{I}(t) s n^2 \hat{\mathbf{s}} \quad \text{[1pt]} \end{aligned}$$

(As you would expect, this points radially inwards if $\dot{I}(t) > 0$; the solenoids feeds energy to its interior to strengthen the field there.)

7. (3 points) Show/argue that the fields given above violate local energy conservation if $\ddot{I}(t) \neq 0$. Your choice which region you consider.

Solution:

Method 1: $s > R$, using words

- There is no magnetic field, so the Poynting vector is zero and the energy density is $u = \epsilon_0 E^2/2$ [1pt].
- If $\dot{I}$ changes with time, **E** and therefore u changes with time [1pt].
- But if there is no energy flux (as encoded by the Poynting vector), this contradicts energy conservation. [1pt]

Method 2: $s > R$, using math Energy conservation requires

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{S}$$

(since there are no charges to do work on outside the solenoid) [1pt]. Since there is no magnetic field, $\mathbf{S} = 0$ and $u = \epsilon_0 E^2/2$ [1pt]. Thus

$$\begin{aligned} 0 = \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{\epsilon_0 E^2}{2}\right] \quad [1\text{pt}] \\ &= \epsilon_0 E \frac{\partial E}{\partial t} = \frac{\epsilon_0 {\mu_0}^2 \dot{I}(t)\ddot{I}(t) n^2 R^4}{4s^2} \end{aligned}$$

If $\ddot{I}(t) \neq 0$, this is false, as required.

Method 3: $s < R$, using math

Now we need the general $u = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}$ [1pt].

Using the $\mathbf{S}$ from last question, $\nabla \cdot \mathbf{S} = \frac{1}{s}\frac{\partial}{\partial s}(sS_s)$;

$$\frac{\partial}{\partial t}\left[\frac{\epsilon_0}{2}\left[\frac{1}{2}\mu_0 I(t)\dot{I}(t) s n^2\right]^2 + \frac{1}{2\mu_0}\left[\mu_0 I(t) n\right]^2\right] = \frac{1}{s}\frac{\partial}{\partial s}\left[\frac{\mu_0 I(t)\dot{I}(t) s^2 n^2}{2}\right]$$

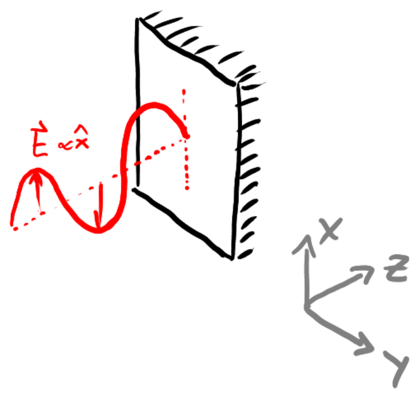
This is false, because the left-hand side will spawn a term $\propto \ddot{I}$ from the derivative acting on $\dot{I}$. There is no matching term on the right-hand side. [1 pt]

(Method 3 was the most popular.)

III. Mirror under pressure [15 points]

A perfect uncharged conductor with $\mu = \mu_0$ and $\epsilon = \epsilon_0$ occupies the half of space with $z > 0$. It allows neither electric nor magnetic fields in its interior. The rest of space is vacuum.

Consider the magnetic field just on the $z < 0$ side of $z = 0$. We will call this $\mathbf{B}(z = 0^-)$. The x and y axes are parallel to the interface as shown.



8. (4 points) By integrating a Maxwell equation, prove that

$$B_{\text{y}}(z = 0^-) = \mu_0 K_x. \tag{5}$$

Here $\mathbf{K}$ is the surface current on the conductor surface.

Solution:

Apply Maxwell-Ampere in integral form:

$$\oint \mathbf{B} \cdot d\ell = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} + \epsilon_0 \mu_0 \frac{d}{dt} \int \mathbf{E} \cdot d\mathbf{a} \quad [1\text{pt}]$$

to a loop of length 1 in the y-direction that is an infinitesimal fraction above and below the surface [1pt, best to draw]. Integrating counterclockwise (as usual) so the K_x surface current has positive orientation, we get

$$\begin{aligned} &B_{\text{y}}(z = 0^-) - 0(\text{no field in conductor}) + 0(\text{from infinitesimal sides}) \quad [1\text{pt}] \\ &= \mu_0 K_x + 0(\text{no flux through infinitesimal area}), \quad [1\text{pt}] \end{aligned}$$

the desired result.

Many people did not choose a loop, and tried to take an indefinite integral (sometimes over time!) of a possibly random Maxwell's equation.

Others did appear to take a correct integral, but either: did not draw the loop (OK but then you have to describe it accurately), did not say why the displacement current vanished (loop has infinitesimal area, no flux through it), or just assumed there was no displacement current (for sure there is in wave fields!).

Some recognized this as a specific case of the more general boundary condition

$$\frac{\mathbf{B}_1^{\|}}{\mu_0} - \frac{\mathbf{B}_2^{\|}}{\mu_0} = \mathbf{K} \times \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the normal to the interface from region 2 to 1. If we take the vacuum as 1, $\widehat{\mathbf{n}} = -\widehat{\mathbf{z}}$. Then the right-hand side becomes

$$K_x(\hat{\mathbf{x}} \times -\widehat{\mathbf{z}}) + K_y(\hat{\mathbf{y}} \times -\widehat{\mathbf{z}}) = K_x\widehat{\mathbf{y}} - K_y\widehat{\mathbf{x}}.$$

Taking the y-component, and nothing $B_2 = 0$ since it is inside the conductor, gives the desired result. Unfortunately, the question asks to show this by integrating a Maxwell's equation – the correct answer basically derives this boundary condition (for this specific case). Thus we give this at most 2 points to answers like this.

A monochromatic electromagnetic plane wave polarized in the x direction propagates along z. It hits the conductor and is perfectly reflected, producing a similar wave propagating along $-z$. Thus, on the vacuum side in our situation,

$$\mathbf{E}_I = E_0 \cos(kz - \omega t)\widehat{\mathbf{x}} \tag{6}$$
$$\mathbf{E}_R = E_0 \cos(-kz - \omega t + \phi)\widehat{\mathbf{x}}. \tag{7}$$

9. (3 points) Explain using a boundary condition that the phase shift ϕ must be π in this case. State exactly what boundary condition you are using **AND** which Maxwell's equation it is derived from (you do not have to do the derivation).

   Hint: what is the consequence of a phase shift of π?

Solution:

- The boundary conditon derived from Faraday's law ...[1pt]
- ...holds that $\mathbf{E}^{\|}$, including E_x, is continuous across the interface [1pt].
- Thus $E_x(z = o^-) = 0$ for all times, which means the waves must cancel there, as happens with a phase shift of π [1pt].

(A π phase shift spawns a minus sign, from $e^{i\pi} = -1$, $\cos(u \pm \pi) = -\cos(u)$, or drawing a wave.)

Common mistakes: attempting to use a different boundary condition, forgetting to state which Maxwell equation it comes from (you could have guessed, there are only four...).

10. (3 points) Show that

$$\mathbf{B}(z=0^-) = \frac{2E_0}{c}\cos(\omega t)\widehat{\mathbf{y}}. \tag{8}$$

Solution: Using $\mathbf{B} = \hat{\mathbf{k}} \times \mathbf{E}/c$, we can find the magnetic fields (at $z = 0^-$):

$$\mathbf{B}_I(z = 0^-) = \frac{E_0}{c}\cos(-\omega t)(\widehat{\mathbf{x}} \times \widehat{\mathbf{z}}) = \frac{E_0}{c}\cos(\omega t)\widehat{\mathbf{y}} \quad [1\text{pt}]$$
$$\mathbf{B}_R(z = 0^-) = -\frac{E_0}{c}\cos(-\omega t)(-\widehat{\mathbf{x}} \times \widehat{\mathbf{z}}) = \frac{E_0}{c}\cos(\omega t)\widehat{\mathbf{y}} \quad [1\text{pt}]$$

where the first minus sign in the second line is due to the π phase shift. Thus:

$$\mathbf{B}(z = 0^-) = \mathbf{B}_I(z = 0^-) + \mathbf{B}_R(z = 0^-) = \frac{2E_0}{c}\cos(\omega t)$$

as desired. [1pt]

You can also add the electric fields, invoke Faraday's law, then integrate over time. This is considerably harder.

Some started by adding the fields and then unsuccessfully throwing various trig identities at it.

On the $z > 0$ of the interface, the magnetic field is zero. (The surface current explains the discontinuity.) The average magnetic field

$$\mathbf{B}_{\text{avg}} = \frac{\mathbf{B}(z = 0^-) + \mathbf{B}(z = 0^+)}{2} = \frac{\mathbf{B}(z = 0^-)}{2} \tag{9}$$

acts on the surface current to give a Lorentz force.

11. (5 points) Show explicitly that the time-averaged Lorentz force on the surface current equals twice the time-averaged radiation pressure in the incident wave (as it should, by conservation of momentum).

Solution: The average radiation pressure due to the incident wave is

$$\langle p_z \rangle = \frac{1}{2}\epsilon_0 E_0. \quad [1\text{pt}]$$

The Lorentz force (per unit area) is purely z (since B is purely y and **K** purely x):

$$\begin{aligned} F_z &= K_x B_{\text{avg,y}} \quad [1\text{pt}] \\ &= \frac{K_x B(z{=}0^-)_y}{2} \\ &= \frac{\left(B(z{=}0^-)_y\right)^2}{2\mu_0} \\ &= \frac{2{E_0}^2\cos^2(\omega t)}{\mu_0 c^2} \quad [1\text{pt}] \\ &= 2{E_0}^2\epsilon_0\cos^2\omega t \quad \text{(by definition of c)} \quad [1\text{pt}] \end{aligned}$$

and since $\langle \cos^2\omega t\rangle = 1/2$ [1pt],

$$\langle F_z \rangle = {E_0}^2\epsilon_0 = 2\langle p_z \rangle \tag{10}$$

as required.

The question should of course have asked for the time-averaged Lorentz force *per unit area*, since pressure is a force per unit area. In consideration of this we made one point of this question a bonus point (so you only need 37 points to get a 10).

When you are finished: Write your name and student number on your solutions – both sheets if you used two. Place solutions in the box with your student number. Add a mark next to your name on the list in that box. Return the formula sheet and unused paper to their corresponding stacks. Take the piece of paper you are currently reading home. Exit the hall in the back, not where you entered.