Name: Student Number:

Mechanics and Relativity: M3

Mock exam

Duration: 120 mins

Before you start, read the following:

- There are 3 problems with subquestions, and you can earn 90 points in total. Your final grade is 1+(points)/10.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations:

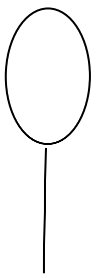
$$\vec{F} = m\vec{a}\,, \quad \vec{L} = \vec{r} \times \vec{p}, \quad \vec{\tau} = \vec{r} \times \vec{F}\,, \quad \vec{F}_{\text{centr}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})\,, \quad \vec{F}_{\text{Cor}} = -2m\vec{\omega} \times \vec{v}\,, \quad \vec{F}_{\text{azim}} = -m(\frac{d\vec{\omega}}{dt}) \times \vec{r}\,,$$

and the Taylor expansion $(1 + ax)^b \approx 1 + abx + \ldots$ at small x.

Question 1: Principal axes

(a) **(10 pts)** Give one of the equivalent definitions of principal axes; in other words, when is an axis principal?

A direction is a principal axis with respect to a certain point if an object can rotate around this axis at constant angular velocity without the need for a torque with respect to this point. OR
In the basis provided by the principal axes, the inertia tensor is a diagonal matrix.



(b) **(10 pts)** Imagine you are given an infinitely thin sheet of material that has an irregular planar shape (so not a circle or rectangle, but some shape without symmetry axis). Pick an arbitrary point on this arbitrary shape. How many principal axes w.r.t. this point lie in the plane of the sheet? Briefly explain your answer.

There are two principal axes in the plane of the sheet. The reason is that the third one is always orthogonal to the plane, this is always a principal axis for a sheet-like 2D object (as it cannot exert a torque). The other two principal axes have to be orthogonal to the third, and hence lie in the plane of the sheet.

(c) **(10 pts)** Draw the principal axis of a tennis racket in order of ascending principal moments.

For the tennis racket as indicated above, the smallest principal moment is along the axis orthogonal to the page, then the left-right axis, then the up-down axis.

Question 2: Spinning chandelier

(a) **(15 pts)** Instead of a spinning top, consider a spinning chandelier that is suspended from the ceiling, and makes an angle θ with the vertical direction. Imagine it has principal moments $I_1 = I_2 \equiv I < I_3$, with the third principal axis along the spinning direction that is vertical when in rest. What is its precession frequency Ω in terms of its spinning frequency ω_3? You can use that the precession frequency Ω for a spinning top instead is

$$\Omega = \frac{I_3\omega_3}{2I\cos(\theta)}\left(1 - \sqrt{1 - \frac{4MIg\ell\cos(\theta)}{I_3^2\omega_3^2}}\right), \tag{1}$$

and it can help to think about the differences in geometrical configurations of both set-ups (that is, top and chandelier).

The difference in geometrical set-ups between the two cases is that for the top, θ is defined by the standing top with respect to the vertical axis, while for the chandelier θ is defined by the hanging chandelier. This is the only difference in the situation. Therefore one can map the top result to that of the chandelier by $\theta \to \theta + \pi$. This maps the case of above the table to the case of below the ceiling. Under this shift, the cosine flips sign, and hence the frequency is

$$\Omega = \frac{I_3\omega_3}{2I\cos(\theta)}\left(\sqrt{1 + \frac{4MIg\ell\cos(\theta)}{I_3^2\omega_3^2}} - 1\right). \tag{2}$$

(Of course this would also follow from a calculation of the torque and resulting rate of change of angular momentum, similar to the derivation of the precession frequency for the top.)

(b) **(15 pts)** Can the chandelier precess (at constant, non-vanishing angle θ) if it does not spin around its x_3 principal axis, that is when ω_3 vanishes? Briefly explain your answer using the no-spinning limit of the precession frequency (if you did not find an answer under a, use the above expression for the spinning top instead). If your answer is yes: give the precession frequency.

Yes, this is possible. Physically, one can understand this as the chandelier is just like any other non-spinning object, e.g. a stick, and these can also precess while hanging down (see the discussion in sec 9.4.2). One can see this mathematically by taking the $\omega_3 \to 0$ limit. This results in

$$\Omega = \sqrt{\frac{Mg\ell}{I\cos(\theta)}}. \tag{3}$$

Question 3: Tidal forces

(a) **(15 pts)** Suppose we would consider a drop of water on Earth that is located along the axis of the Earth and the Moon. Derive what the tidal force (due to the Moon and as measured in the Earth's frame) on this drop of water would be. You can use the approximation of three point objects, $m_{1,2,3}$, on a straight line, with a distance x between m_2 (Earth) and m_1 (drop of water) that is much smaller than the distance R between the Earth and the moon (m_3). Does the tidal force point away or towards the Earth?

See the discussion and result in Morin, sec. 10.3 and eq. (10.28). It points away from the Earth.

(b) **(15 pts)** Next, consider an imaginary Universe where the attraction between these celestial bodies is not due to gravity, with its $1/r^2$ fall-off, but instead the Earth and the Moon experience an attractive force that is of the form of Hooke's law, with $\vec{F} = -k\vec{r}$. In which direction does the tidal force point in such a Universe? Briefly explain your answer, based on physical reasoning / general considerations, not necessarily the full calculation.

With a force that is governed by the Hooke law, the force increases with distance (instead of gravity, which falls off with distance). Therefore, when the water is on the Moon side of the surface of the Earth (so closer than the Earth itself), the Earth will be more attracted by the Moon than the water is, and hence will accelerate faster towards the Moon than the water. Hence the (longitudinal) tidal force in this case is attractive, and points towards the Earth.