Name: Student Number:

Mechanics and Relativity: M resit

February 15, 2024, Aletta Jacobshal
Duration: 120 mins

Before you start, read the following:

- There are 3 problems with subquestions, and you can earn 90 points in total. Your final grade is 1+(points)/10.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations:

$$\vec{F} = m\vec{a}\,, \quad K = \tfrac{1}{2}mv^2\,, \quad V = \tfrac{1}{2}kx^2\,, \quad I = \int x^2 dm\,, \quad \tau = I\alpha\,, \quad \vec{F}_{\rm Cor} = -2m\vec{\omega}\times\vec{v}\,,$$

and the Taylor expansions at small x:

$$(1+ax)^b \approx 1 + abx + \ldots\,, \quad \sin(x) \approx x + \ldots \quad \cos(x) \approx 1 - \tfrac{1}{2}x^2 + \ldots\,. \tag{1}$$

Question 1: Harmonic oscillator

Consider a cart of mass m that can move back and forth on an air rail, such that you can ignore friction in this question. It is attached to a wall via a spring with spring constant k. The resulting oscillation has an eigenfrequency given by $\omega^2 = k/m$.

(a) **(10 pts)** Show with an explicit calculation that the total energy, consisting of kinetic and potential energy, is constant throughout this motion.

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2\,,$$

$$\frac{dE}{dt} = mva + kvx = v(ma + kx)\,,$$

which vanishes due to Newton's second law.
Alternative approach: the general motion is $x = A\cos(\omega t + \varphi_0)$, and then one can calculate K and V, which are separately not constant (but instead $\cos^2$ and $\sin^2$), but add up to a constant.
(10pts for either of these proofs; partial: 5pts for calculating time derivative of E, or 5pts for calculating explicit expressions of K and T)

(b) **(10 pts)** Now add a driving force of the form $F_0 \cos(\omega_d t)$. What is the amplitude of the resulting driven motion for $\omega_d = \frac{1}{2}\omega$? You can ignore the homogeneous solution, and only focus on the particular solution in this case.

Newton's 2nd law now reads $m\ddot{x} = -kx + F_0 \cos(\omega_d t)$. The particular solution will follow the time-dependence of the driving force, and hence is of the form $x = A\cos(\omega_d t) + B\sin(\omega_d t)$. Plugging this in gives $B = 0$ and

$$-m\omega_d^2 A = -kA + F_0 \,, \rightarrow A = \frac{F_0}{m(\omega^2 - \omega_d^2)} = \frac{4F_0}{3m\omega^2} \,. \tag{2}$$

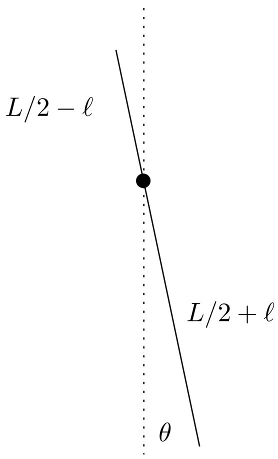
(10pts for correct answer. Partial points: 5pts for x in terms of $\sin$ and $\cos$ with $\omega_d t$ dependence.)

(c) **(10 pts)** What is the phase difference between the driving force and the driven oscillation when $\omega_d < \omega$? And what is it when $\omega_d > \omega$? Remember that you can ignore friction in this exercise.

The phase difference will be zero (in phase) when the driving frequency is smaller than the eigenfrequency, and will be π (out of phase) when it is larger.
(5pts for correct answer per regime)

Question 2: Moment of inertia and angular acceleration

Consider a stick of length L, mass M and uniform density $\rho = M/L$ that is fixed by a pivot point at a distance ℓ from its center of mass; as seen from the pivot point, the upper and lower parts therefore have lengths $L/2 \pm \ell$, see the picture. In this exercise you can restrict to planar motion, with only two spatial coordinates and one angle θ with the vertical direction.



(a) **(15 pts)** Calculate the moment of inertia of the stick with respect to the pivot point.

One approach, using the CM: Around the CM the moment of inertia is given by

$$I_{CM} = 2\int_0^{L/2} \rho x^2 dx = 2(\tfrac{1}{3}\rho(\tfrac{1}{2}L)^3) = \tfrac{1}{12}ML^2\,. \tag{3}$$

The pivot point is a distance ℓ away from this. Using the parallel axes theorem,

$$I = I_{CM} + M\ell^2\,. \tag{4}$$

Alternatively, one can calculate this answer also by doing the full integral,

$$I = \int_{-L/2+\ell}^{L/2+\ell} \rho x^2 dx \tag{5}$$

(15pts for correct answer; partial points: 5pts for correct formula for I or 5pts for parallel axes relation.)

(b) **(15 pts)** Calculate the magnitude of the torque due to gravity as a function of ℓ and the angle θ. Moreover, calculate the angular frequency ω (as a function of ℓ and of the moment of inertia I) of the resulting oscillatory motion of the stick under the influence of gravity. You can use the approximation that the angle θ is small.

Gravity acts downwards with a strength $F_g = -Mg$. The horizontal part of the arm has a length $\ell\sin(\theta) \approx \ell\theta$. The torque is therefore $-Mg\ell\theta$. Using $\tau = I\alpha$, we have

$$-Mg\ell\theta = I\ddot{\theta}\,. \tag{6}$$

This results in oscillations with angular frequency $\omega = \sqrt{Mg\ell/I}$.
(7pts for correct torque (3pts for force, 4pts for correct arm); 8pts for correct angular frequency (4pts for relation between torque τ and angular acceleration $\alpha = \ddot{\theta}$, and 4pts for solving for ω.)

Question 3: Foucault's pendulum

Imagine a pendulum consisting of a heavy mass M hanging from a rope of length L at the North pole. For small oscillations, the time it takes for the pendulum to swing from left to right under the influence of gravity is $T = \pi\sqrt{L/g}$. The pendulum is so long that this takes four minutes. Use coordinates (x, y) for the surface at the North pole.

(a) **(10 pts)** The Coriolis force causes a small deflection what will make the pendulum precess. Consider a single swing from left to right, along the x direction: $x(t) = -x_0 \cos(\pi t/T)$. During this motion, what is the magnitude of the Coriolis force as a function of time? Sketch the motion that this leads to in the (x, y) plane, as the pendulum swings from left to right.

Magnitude of Coriolis force: $F = 2m\omega v = 2m\omega x_0\pi/T \cdot \sin(\pi t/T)$.
Motion: as the pendulum swings from $-x_0$ to x_0, it starts out at $y = 0$ and moves a bit to negative y (as seen from the pendulum, the Coriolis force acts to the right on the Northern hemisphere).
(5pts for correct magnitude, 5pts for correct sketch / description of motion.)

(b) **(20 pts)** As the pendulum swings from left to right along the x-axis, what is the velocity and the position (as a function of time) along the y-axis induced by the Coriolis force? How large is the deflection along the y-axis that this leads to during one swing from left to right? By how many degrees has the pendulum precessed during these four minutes? Remember that $\omega = 7{\cdot}10^{-5}$ 1/s, take x_0 to be 1 m and only calculate to one significant digit.

The force above gives the acceleration in the y-direction:

$$\ddot{y} = -2\omega x_0\pi/T \cdot \sin(\pi t/T)\,. \tag{7}$$

The velocity comes from integrating once, and is

$$\dot{y} = 2\omega x_0(\cos(\pi t/T) - 1\,, \tag{8}$$

with the constant of integration chosen such that it starts out with zero velocity. Then the position is

$$y = 2\omega x_0(T/\pi \cdot \sin(\pi t/T) - t) \tag{9}$$

This starts at $y = 0$ at the beginning of the swing (as it should), and becomes at the end $y = -2\omega x_0 T$. Putting in the numbers gives $y_0 = -3$ mm. This corresponds to a precession of $\arctan(0.03/2) \approx 1$ degree.
(5pts for correct velocity (3pts for time-dependence, 2pts for integration constant), 5pts for correct position (3pts for time-dependence, 2pts for integration constant), 5pts for numerical value of deflection, 5pts for correct precession angle (also if this is calculated by using the 360 degree rotation in 24hrs))