

Mechanics and Relativity: M3

January 24, 2023, Aletta Jacobshal

Duration: 120 mins

Before you start, read the following:

- There are 3 problems with subquestions, and you can earn 90 points in total. Your final grade is $1 + (\text{points})/10$.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations:

$$\vec{F} = m\vec{a}, \quad \vec{L} = \vec{r} \times \vec{p}, \quad \vec{\tau} = \vec{r} \times \vec{F}, \quad \vec{F}_{\text{centr}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}), \quad \vec{F}_{\text{Cor}} = -2m\vec{\omega} \times \vec{v}, \quad \vec{F}_{\text{azim}} = -m\left(\frac{d\vec{\omega}}{dt}\right) \times \vec{r},$$

and the Taylor expansion $(1 + ax)^b \approx 1 + abx + \dots$ at small x .

Question 1: Principal axes

- (a) **(10 pts)** According to Chasles' theorem, how does one characterize the most general behaviour (at a given moment) of a solid object with reference to an arbitrary point P inside that body? Which vector quantities does this require?

This theorem states that the most general motion of a rigid body (at a given moment) consists of linear motion of any point in that body, combined with rotation around that point. This therefore requires the linear velocity $\vec{v}$ and the angular velocity $\vec{\omega}$.

(5pts for combination of linear motion plus rotation, 5pts for linear plus angular velocity)

Name:

Student Number:

- (b) **(10 pts)** Consider an object that has the shape and orientation of a door, and assume its density to be constant. One of the principal axes with respect to the door's center of mass is the vertical direction. Is this also a principal axis with respect to other points of the door that lie along this vertical direction? Briefly explain your answer.

Yes, this will also be a principal axis with respect to other points along the line. Due to the left-right symmetry of the door, the torques generated by the centrifugal forces (which would cause the door to rotate around a different axis) cancel out, and hence the door is rotating around a principal axis.

(5pts for correct answer, 5pts for argument along the lines of cancellation of centrifugal forces and torques.)

- (c) **(10 pts)** Now instead imagine a random shape in 3D, e.g. a potato-like object with no symmetries. Pick a random plane that goes through the center of mass of the object. How many principal axes of the object with respect to the center of mass would you expect to lie in this plane? Briefly explain your answer.

For a random object, there will be zero principal axes in a given frame. The reason for this is that the principal axes are fixed by the object, and hence are a fixed coordinate system. A randomly given plane will not lie along any of these coordinate systems. (If you explain that, depending on the choice of the plane, it can range from 0 (the generic case) to 2 (the special case), that is fine as well.)

(5pts for correct answer, 5pts for reasoning on principal axes and given plane.)

Question 2: Spinning top without fixed pivot point on table

Consider a spinning top that is spinning and precessing at a table, without its bottom point being fixed to the table - instead, the bottom point can move freely and without friction. Take the top to be spinning clockwise (as seen from above). It has a center of mass at a distance ℓ from its bottom point, and makes an angle θ with respect to the vertical direction. Its moment of inertia around the spinning direction is I_3 , and it is spinning with angular frequency ω_3 . Throughout this exercise, you can use the fast-spinning approximation (in which angular momentum is dominated by spinning) and you only have to consider vertical forces.

- (a) **(15 pts)** With respect to the center of mass of the spinning top, which force(s) induce(s) a torque? What is the magnitude and direction of this torque? Moreover, what is the magnitude and direction of angular momentum? Indicate in a planar drawing in which direction the horizontal component of angular momentum and its rate of change are pointing.

The normal force, with magnitude $N = mg$ and acting upwards on the bottom part of the spinning top, induces a torque. The magnitude of the torque is $\ell N \sin(\theta)$, and its direction is horizontal: when the top is leaning over to the left, the torque points downwards (as seen from above). The angular momentum points along the direction of the spinning top in the downward direction, and its magnitude is $\omega_3 I_3$. Projected onto the horizontal plane, it therefore points to the right (with magnitude $\omega_3 I_3 \sin(\theta)$). The rate of change of angular momentum is given by the torque, and hence points downwards in the plane.

(5pts for correct force, 5pts for correct torque, 5pts for correct angular momentum (of which 2 for magnitude and 3 for direction))

- (b) **(15 pts)** The bottom point of the top will trace out a circle on the table. Derive the radius, frequency and orientation of this motion.

The radius will be $\ell \sin \theta$ (from geometric set-up: the center of mass is fixed), and the orientation will be clockwise (this follows from the rate of change as indicated upwards, and also from the general discussion: the orientation of the spinning and precession is always the same, so clock-wise in this case. The frequency of this motion follows from the magnitude of horizontal angular momentum and its rate of change (as discussed in Morin and during the lecture), and thus becomes

$$\Omega = \frac{Mg\ell}{I_3\omega_3}. \quad (1)$$

(3pts for correct radius, 3pts for correct orientation, 9pts for correct frequency.)

Question 3: Dropping a ball

Imagine you live somewhere on earth at a latitude of 45 degrees, so exactly in between the North Pole and the equator. Approximate the Earth to have a radius of 6400 km, an angular speed of $\omega = 7 \cdot 10^{-5}$ 1/s and a gravitational acceleration of $g = 10$ m/s². We will be dropping a ball from a height of five meters, resulting in a drop time of one second.

- (a) **(15 pts)** We will first consider the West-East deflection. Which fictitious force is responsible for this? Calculate the deflection (one significant digit) and indicate its direction.

This is the Coriolis force. It will deflect the ball in the Eastern direction. The magnitude of this force is given by $2m\omega v$. For a ball dropping with vertical velocity gt , this implies $x = 1/3\omega gt^3 \sin(\theta)$. It results in 0.2 mm.

(3ts for correct force, 3pts for correct direction, 9pts for correct magnitude of deflection (of which 3 for force, 3 for general formula for deflection, 3 for numerical value.)

- (b) **(15 pts)** Now we turn to the North-South deflection. Which fictitious force is responsible for this? Calculate the deflection (one significant digit) and indicate its direction.

This is the centrifugal force. It will deflect the ball in the Southern direction. The magnitude of the force is given by $m\omega^2 R \cos(\theta)$, where R is the radius of the Earth. For a ball dropping for one second, this constant acceleration $a = \omega^2 R \cos(\theta)$ results in $y = 1/2at^2$. This results in 1 cm.

(3ts for correct force, 3pts for correct direction, 9pts for correct magnitude of deflection (3 for force, 3 for general formula for deflection, 3 for numerical value.)