Name: Student Number:

Mechanics and Relativity: M1

December 1, 2023, Aletta Jacobshal
Duration: 60 mins

Before you start, read the following:

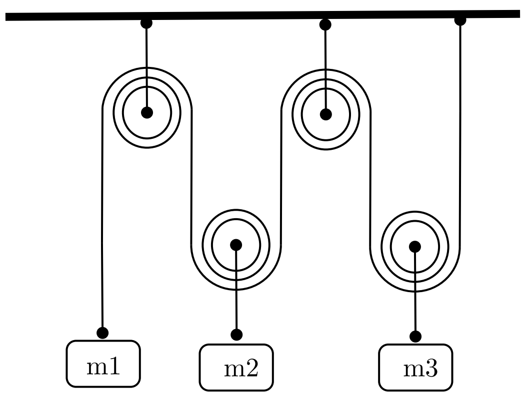
- There are 2 problems with subquestions, and you can earn 90 points in total (45 per problem). Your final grade is 1+(points)/10.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations and values:

$$F = ma\,, \quad E = mc^2\,, K = \frac{1}{2}mv^2\,, \quad V = mgh\,, \quad V = -\frac{1}{2}kx^2\,, \quad g \approx 10 m/s^2\,. \tag{1}$$

Question 1: A double Atwood machine

Consider an extended Atwood machine as indicated in the picture, with a single rope connecting three masses m_i. You can assume the rope is massless and does not stretch, and the pulleys are massless and have no friction.



(a) **(15 pts)** Which ratios of the three masses are needed to achieve a static configuration?

Newton's equations for the three masses:

$$m_1 a_1 = m_1 g - T\,, \quad m_2 a_2 = m_2 g - 2T\,, \quad m_3 a_3 = m_3 g - 2T\,. \tag{2}$$

Note that we have used that the tension is constant throughout the rope (no friction or other forces pointing along the rope). For static configurations, the three right hand sides must all vanish, which requires

$$m_1 = \tfrac{1}{2} m_2 = \tfrac{1}{2} m_3 \tag{3}$$

(15pts for correct ratios - partial: 5pt for correct Newton's eqs, 5pt for accelerations equal to zero.)

(b) **(15 pts)** Which acceleration does m_1 have when all three masses are equal?

All masses equal implies (from the last two Newton's equations) that $a_2 = a_3$. Moreover, we have from the rope configuration that $y_1 = -2y_2 - 2y_3$ and hence also $a_1 = -2a_2 - 2a_3$. Plugging this into the above two independent equations and solving for the two unknowns (a_1 and T) one finds

$$a_1 = \tfrac{4}{9} g\,. \tag{4}$$

(15pts for correct acceleration - partial: 5pt for conservation of string / right relation between y's, 5pt for $a_2 = a_3$)

(c) **(15 pt)** Is the combination of potential energy and kinetic energy of the first mass m_1 conserved when it accelerates as in (b) (so with all masses equal)? Show how this follows from a calculation. (If you did not find an answer at (b), take the acceleration to be half of the gravitational one, i.e. $a = g/2$.) Explain in one or two sentences whether this combination of energy should be conserved.

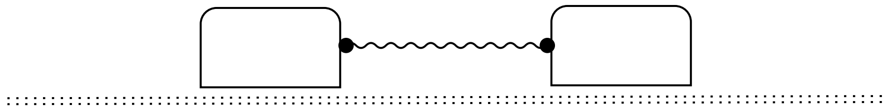
Not, it is not constant. You can solve for the motion of the first block: its acceleration is $(4/9)g$, and hence $y_1 = \text{constant} - (2/9)gt^2$. This implies that the kinetic energy is $K = 1/2m(4/9)^2 g^2 t^2$ while potential energy is $V = mgy_1$. The sum of these is not time-independent. This makes sense as energy is transferred into the other two blocks as they are pulled upwards (and hence their kinetic and potential energies increase).

(5pts for No, 5pts for calculation, 5pts for explanation)

Name: Student Number:

Question 2: Normal modes of two carts

Consider rails (of infinite extent) with two carts (of equal masses m) that are connected by a spring with spring constant k, see the picture.



(a) **(15 pts)** What are Newton's 2nd laws for the two carts, expressed in terms of their locations $x_{1,2}$? Give the explicit form of these differential equations for the first cart and for the second cart.

Newton's 2nd laws:

$$m\ddot{x}_1 = -k(x_1 - x_2)\,, \qquad m\ddot{x}_2 = -k(x_2 - x_1)\,. \tag{5}$$

(15pts for correct answer - partial: 7.5pts for correct force on x_1 due to x_1 (and 2), 7.5pts for correct force on x_1 due to x)

(b) **(15 pts)** Derive what are the normal modes (as linear combinations of the locations $x_{1,2}$ of the two carts) of this system and the differential equations governing their dynamics. Moreover, what is the most general solution of this system? *Hint: when expressed in terms of normal modes, Newton's 2nd laws become two separate ODE's for the two combinations.*

By taking the sum and difference of the above equations, we find

$$m(\ddot{x}_1 + \ddot{x}_2) = 0\,, \quad m(\ddot{x}_1 - \ddot{x}_2) = -2k(x_1 - x_2)\,, \tag{6}$$

and hence the normal modes are $x_1 \pm x_2$. The most general solution in this case is

$$x_1 + x_2 = A + Bt\,, \quad x_1 - x_2 = C\cos(\omega t) + D\sin(\omega t)\,, \tag{7}$$

where $\omega^2 = 2k/m$.

(5pts for correct normal modes $x_1 \pm x_2$, 5pts for correct ODEs, 5pts for correct solutions)

(c) **(15 pts)** Now include a friction term due to e.g. a viscous fluid in which the system is immersed. It is modelled by a drag force $-2m\gamma\dot{x}_{1,2}$ (with γ constant) that is proportional to the velocities of the carts. How does this change the differential equations governing the normal modes? What is the solution in this case (with a small friction term, i.e. the underdamped case)?

Newton's laws:

$$m\ddot{x}_1 + 2m\gamma\dot{x}_1 + k(x_1 - x_2) = 0\,, \qquad m\ddot{x}_2 + 2m\gamma\dot{x}_2 + k(x_2 - x_1) = 0\,, \tag{8}$$

The normal modes are the same linear combinations, and both have a damping term. The solution for the previously uniform motion now becomes

$$x_1 + x_2 = A + Be^{-2\gamma t}\,, \tag{9}$$

and for the other one

$$x_1 - x_2 = e^{-\gamma t}(C\cos(\tilde{\omega}t) + D\sin(\tilde{\omega}t))\,, \tag{10}$$

where $\tilde{\omega}^2 = \omega^2 - \gamma^2$.

(5pts for correct Newton's eqs, 5pts for correct damped uniform sol, 5pts for correct damped oscillation dol (of which 3 for oscillation times damping, and 2 for correct frequency))