Mechanics and Relativity: M2

December 19, 2023, Aletta Jacobshal
Duration: 90 mins

Before you start, read the following:

- There are 3 problems with subquestions, and you can earn 90 points in total. Your final grade is 1+(points)/10.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

Possibly relevant equations and values:

$$F = ma\,, \quad \vec{L} = \vec{r} \times \vec{p}\,, \quad T = \tfrac{1}{2} I_z \omega^2\,, \quad \tau = I\alpha\,. \tag{1}$$

Question 1: Central forces

(a) **(10 pts)** What is the requirement for a force to be central? You can phrase this requirement either in terms of the force $\vec{F}$ or its potential energy V (either is fine).

For a force to be central one requires that its direction is radial and its magnitude only depends on the radius, $\vec{F} = F(r)\hat{r}$. This corresponds to the potential energy only depending on the radius, $V = V(r)$.
(10pts for correct requirement (either force or potential) - partial: only direction / magnitude condition on force: 5pts)

(b) **(10 pts)** Prove that a central force preserves angular momentum.

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\vec{r} \times \vec{p} = \vec{v} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F}\,, \tag{2}$$

and the two terms in the final expression vanish as the two vectors are parallel (linear momentum and velocity are always parallel, and the force and position are parallel for central forces).
(2pts for applying chain rule correctly, 2pts for using Newton's law, 2+4pts for parallel argument)

(c) **(10 pts)** A prominent example of a central force is Newton's law of gravity that governs the attraction between the Sun and other celestial bodies. The resulting trajectories include ellipses and hyperbola. What is the difference between these types of trajectories in terms of their total energy; in other words, how can one distinguish these based on their energy?

The energy corresponding to a hyperbola is positive while that for an ellipse is negative. Or: the energy of a hyperbola is larger than that of an ellipse.
(10pts for either of the above arguments.)

Question 2: Moment of inertia

(a) **(15 pts)** Calculate the moment of inertia of a massive ring of total mass M and radius R, where all mass is located at the rim. Consider the case where the ring lies in the horizontal plane and the axis of rotation is vertical and goes through the center of mass.

In this configuration, the mass of an infinitesimal piece of the ring is $dm = \rho R d\theta$. Integrating over this gives the total mass, $M = 2\pi\rho R$. Then the moment of inertia is given by

$$I = \int r^2 dm = \int_0^{2\pi} R^2 \rho R d\theta = 2\pi\rho R^3 = MR^2 \,, \tag{3}$$

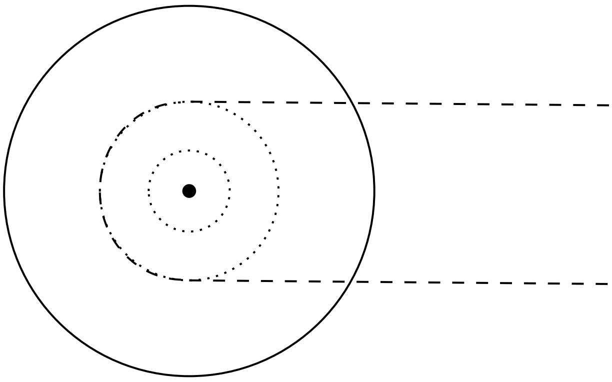
(5pts for mass element dm, 5pts for integrand $r^2 dm$, 5pts for correct answer MR^2 in terms of M)

(b) **(10 pts)** Now consider the same set-up, but now the axis of rotation goes through the outer edge of the ring (and it still vertical). What is the difference between the moments of inertia of this set-up and that of question (a)? (You can answer this question also when you didn't get an answer at the previous question.)

Parallel axes theorem: the two moments of inertia differ by an amount MR^2, where the moment of inertia around the CM is smaller and the current one is larger.
(5pts for correct difference MR^2, 5pts for which one is larger; partial: 5pts for parallel axes theorem)

Question 3: Bikes and gears

Consider a bike with different gears on the rear wheel. For simplicity we will take the rear wheel to have a radius R, and the bike chain can go via either a gear wheel of radius $R/2$ or one of $R/4$. In the picture, the use of the gear wheel of radius $R/2$ is illustrated.



(a) **(10 pts)** Suppose a person is exerting a fixed force on the chain (via the pedals). Which of the two gear wheels will lead to the largest acceleration of the bike, the one of radius $R/2$ or $R/4$, or are they the same? If different, indicate the ratio of the accelerations in the two cases. Briefly explain your answer using the notion of torque.

The larger gear wheel of radius $R/2$ will lead to the largest acceleration. The reason is that this leads to the the largest torque (the force is the same, and the lever arm is twice as large for this gear as compared to the other one) and hence the angular acceleration will also be larger (in fact double).
(5pts for correct answer, 5pts for reasoning that the torque increases as the distance r increases while F is constant)

(b) **(10 pts)** Suppose the acceleration remains constant as the person speeds up from 0 to 30 km/h. How does the work exerted by the person compare in the two cases, using the different gear wheels - which one is larger, or are they the same? Briefly explain your answer.

The work in both cases will be identical. The reason for this is that the total system is conservative (ignoring friction), and hence the two resulting situations carry the same energy, and hence require the same amount of work. (Alternative reasoning: the work is the product of force times distance, which is this case is the length of chain that is pedalled forward, which is the same in both cases - but the latter point is not so easy to see).
(5pts for correct answer, 5pts for conservation of total energy implying work has to be identical; partial: 2pts for work is force times distance.)

(c) **(15 pts)** Express the total energy of the moving bike in terms of its velocity v, its mass M and the wheel radius R. Use the approximation that half of the mass is in the frame of the bike plus the driver, and the other half is in the masses of the wheels (all at the rim), i.e. either wheel carries 25 percent of the mass. You can use your answer under question (2a) here (if you didn't get an answer there, you can use $I = \frac{4}{9}MR^2$ as an answer for question (2a)).

Due to the additive nature of kinetic energy, we have that the total energy (there is no potential energy) is given by the translational motion of the CM, given by $\frac{1}{2}Mv^2$, and the rotational motion, which is per wheel $\frac{1}{2}I\omega^2$. Moreover, we need to use that $v = \omega R$, and $I = \frac{1}{4}MR^2$ - as this is the moment of inertia of a ring of mass $M/4$. Putting this together results in

$$K = K_{\text{transl}} + K_{\text{rot}} = \frac{1}{2}Mv^2 + 2 \times \frac{1}{2}\frac{1}{4}Mv^2 = \frac{3}{4}Mv^2 \tag{4}$$

(15pts for correct answer - partial: 5pts for correct additive expression, 5pts for correct rotational energy per wheel)