Electricity and Magnetism Test 5

23 May 2025, 18:30-20:30

- You may use your double-sided A4 cheat sheet, the provided formula sheet, and a calculator.
- Please leave margins for grading. Do *not* use this paper, or the white scratch paper, for final answers.
- Clearly indicate directions of vector quantities. Make or copy diagrams if it helps you.
- The maximum score is 30 points. Good luck!

Solution:

If you handed in the test, your grade will be:

$$\min\left(10,\ \ 1 + 9 \cdot \frac{\text{Total points}}{30} + \text{Tutorial bonus (if applicable)} \right).$$

Instructions for grading TAs:

- If several students make a similar mistake, please write that down AND agree on a consistent way to score it!
- Only award full integer points equal or larger than zero. Exception: sometimes the rubric says half point may be awarded for a specific circumstance. If you think we need a new case for 0.5pt, agree with other TAs on a rule and write that down (so we can update the rubric).
- If the question asks for an explanation, calculation, determination, argumentation, etc., award no points if this is missing or clearly incorrect / incoherent.
- Subtract points for each mistake only once, unless the error substantially simplifies or alters the rest of the problem.
- Pay close attention to answers for 'show that ...' questions. Making two mistakes that miraculously cancel each other should be awarded *fewer* points than making one mistake and not reaching the result.

I. Short questions [12p]

1. Consider the function $f(x, t) = A\sin(kx)\cos(kvt)$.

   (a) (2 points) Show that this is a solution to the one-dimensional wave equation with wave velocity v and distance x along the medium.

Solution: The wave equation is

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}. \quad \text{[1 pt]}$$

To determine if the given equation is a solution, we simply substitute in the function:

$$\frac{\partial^2 f}{\partial t^2} = -k^2 v^2 A \sin(kx)\cos(kvt)$$
$$\frac{\partial^2 f}{\partial x^2} = -k^2 A \sin(kx)\cos(kvt)$$

which are equal up to the required factor v^2 [1pt].

Scoring guide: It is also OK to write $\frac{\partial^2 f}{\partial t^2} = -k^2 v^2 f(x,t)$ and $\frac{\partial^2 f}{\partial x^2} = -k^2 f(x,t)$ for the second point.

If you make a calculation mistake in either derivative, you lose the second point. (Since you know the two must be equal up to v^2, it is easy to detect your mistake or even work towards the correct answer in the second differentiation.)

(b) (1 point) What type of wave does this specific solution represent? Just name it, no explanation needed.

> **Solution:** This is a (harmonic, one-dimensional) *standing wave.* [1pt]
> 'Harmonic wave' is not enough, we asked what this *specific* solution represents. There are also harmonic traveling waves like $\sin(kx \pm \omega t)$. 'Monochromatic' is similarly not precise enough (two traveling monochromatic waves are needed to make this wave).

2. Consider red and blue light in a non-magnetic linear medium whose permittivity increases with frequency.

(a) (1 point) State which color has higher frequency. No explanation needed.

> **Solution:** The blue light beam has a higher frequency. [1pt]
> (This is a simple empirical fact which you should know. There is no reason for it, we just call visible light with fairly high frequency, which stimulates certain receptors in our eyes etc. 'blue'.)

(b) (2 points) Based on this, compare the **speed** (phase velocity) and **wavelength** of the two colors. Explain which color has the lower/higher value, or why there is no difference.

> **Solution:** $v \propto 1/n \propto 1/\sqrt{\epsilon}$. Since blue light has a higher frequency, ϵ is higher for it, so blue light moves slower. [1pt]
> Since $\lambda = v/f$, and blue light has both a higher frequency and lower v, it has a shorter wavelength. [1pt]
> **Scoring guide**:
>
> - Do not award any points for a correct conclusion without a correct argument.
> - If you start by saying v is equal for both colors, you can earn a maximum of 0.5 points for a consistent continuation (substantial simplification).
> - If in (a) a student answered that blue light has a *lower* frequency than red, they should reach the reverse conclusion here. Award no points for 'accidentally' reaching the correct conclusion through another error. Make sure to grade 2a and 2b at the same time for each student.
> - If the conclusion for v is reversed, the consistent conclusion for λ is that it is not possible to determine which is higher or lower (since it depends on *how* the permittivity changes with frequency). This should probably clue you in that you made a mistake, but a clear explanation of this suffices for 1pt.
> - If in (a) a student says both colors have the same frequency (in defiance of the question's formulation), award no points for 'consistently' arguing that there is no difference in v and λ (substantial simplification; how do you think colors of light differ?).

3. (3 points) Briefly define/explain the terms **plane of incidence**, **s-polarization**, and **Brewster's angle** (explain what it means, not how to calculate it).

> **Solution:**
>
> 1. The plane of incidence is the plane that contains the incident wave vector $\mathbf{k}_I$, the normal to the surface, the reflected wavevector $\mathbf{k}_R$, and the transmitted wavevector $\mathbf{k}_T$. [1pt]
> 2. An s-polarized wave has an electric field vector that oscillates perpendicular to the plane of incidence. [1pt]
> 3. Brewster's angle is the angle of incidence at which p-polarized light is purely transmitted / not reflected. [1pt]
>
> **Scoring guide / common mistakes**:

1. Only two of the vectors in the plane of incidence need be mentioned. Instead of wavevectors you could also mention the directions of the rays, or the directions of propagation. No points for speaking of 'the direction of the wave' – this is ambiguous since for transverse waves like electromagnetic waves, the direction of oscillation and propagation are different.
2. Some examples that don't earn the point: Saying s-polarized light is polarized perpendicular to the wavector (that's true for all electromagnetic waves) or to "the page" (true if the page is the plane of incidence, which is the sensible choice, but not all pages in the universe are aligned the same way). Defining s-polarized light as polarized parallel to the surface is also not worth the point – this is true for all electromagnetic waves at normal incidence. Saying s-polarization is a legal direction of polarization orthogonal to p-polarization is not worth the point unless you then define what p-polarization means.
3. Forgetting to mention that the pure transmission in Brewster's angle only applies to p-polarization (or equivalent remarks) costs 0.5 point. Similarly if you incorrectly say it is the angle at which s-polarized light is purely transmitted (no such angle exist, but at least you remembered that Brewster's angle is related to pure transmission of one polarization). Zero points for saying Brewster's angle is an angle at which one polarization is purely reflected (it's not). Also zero points for "Brewster's angle is the angle at which $\tan\theta_I = n_2/n_1$ / the incident angle whose tangent equals the ratio of the refractive indices / the angle at which the transmitted and reflected waves are at right angles" – it's nice that you remembered how to calculate it, but you were explicitly asked not to give this as your answer.

4. (3 points) A traveling monochromatic electromagnetic plane wave in vacuum has the magnetic field

$$\tilde{\mathbf{B}}(\mathbf{r},t) = -\tilde{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}} \tag{1}$$

with $\mathbf{k} = (-k, 0, k)/\sqrt{2} = k(\hat{\mathbf{z}} - \hat{\mathbf{x}})/\sqrt{2}$. Give the corresponding expression for the **electric** field $\tilde{\mathbf{E}}(\mathbf{r},t)$ of this wave. Use only quantities that appear on the right-hand side of the equation above, fundamental constants, and standard unit vectors. Do not leave unevaluated cross products.

Solution:

$$\tilde{\mathbf{E}}(\mathbf{r},t) = -B_0 c e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{\hat{\mathbf{x}}+\hat{\mathbf{z}}}{\sqrt{2}} \tag{2}$$

Scoring: one point for correct phase, one point for correct magnitude, one point for correct direction. The solution should mention $\mathbf{B} = \mathbf{E}\times\hat{\mathbf{k}}/c$ is used. Giving the real field ($\cos(\ldots)$ instead of $e^{i\cdots}$) is fine too.

Derivation from $\mathbf{B} = \hat{\mathbf{k}}\times\mathbf{E}/c$:

- The exponentials of the two fields are identical, since the electric and magnetic fields must be in phase. So we just have to determine $\tilde{\mathbf{E}}_0$, the field at the origin.
- The magnitude is immediate, $E_0 = cB_0$.
- There are several ways to compute the direction. We know $\hat{\mathbf{E_0}}$ is orthogonal to $\hat{\mathbf{B_0}} = (0,-1,0)$ and the direction of propagation $\hat{\mathbf{k}} = (-1,0,1)/\sqrt{2}$. The only two candidate directions are $(1,0,1)/\sqrt{2}$ and $(-1,0,-1)/\sqrt{2}$. Let's try the second:

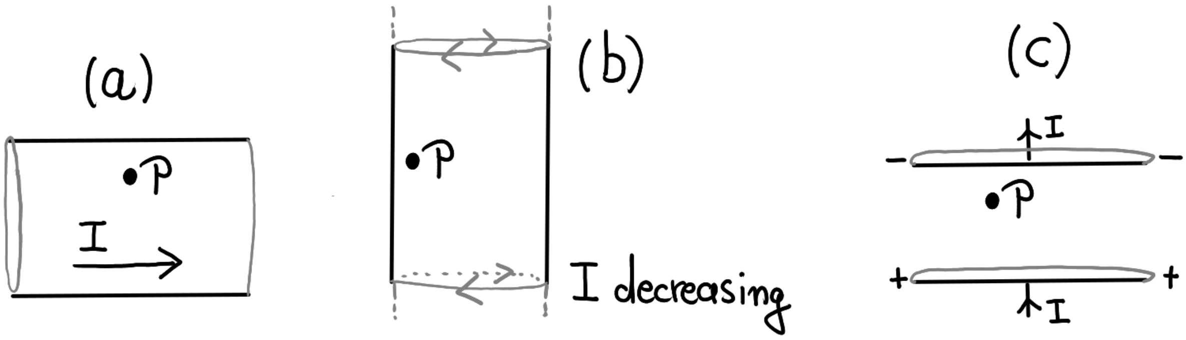
$$\hat{\mathbf{k}}\times\hat{\mathbf{E_0}}/c = \frac{1}{2}(-1,0,1)\times(-1,0,-1) = \frac{1}{2}(-\hat{\mathbf{x}}\times-\hat{\mathbf{z}} + \hat{\mathbf{z}}\times-\hat{\mathbf{x}}) = \hat{\mathbf{x}}\times\hat{\mathbf{z}} = -\hat{\mathbf{y}} \tag{3}$$

which is the given direction for $\mathbf{B}$. The other direction would give a minus sign. You can also check the direction of the Poynting vector:

$$\hat{\mathbf{S}} = \frac{1}{\mu_0}\hat{\mathbf{E}}\times\hat{\mathbf{B}} \propto (-1,0,-1)\times(0,-1,0) = (1,0,-1)\propto\hat{\mathbf{k}}. \tag{4}$$

Finally, you could use the right-hand rule and a drawing.

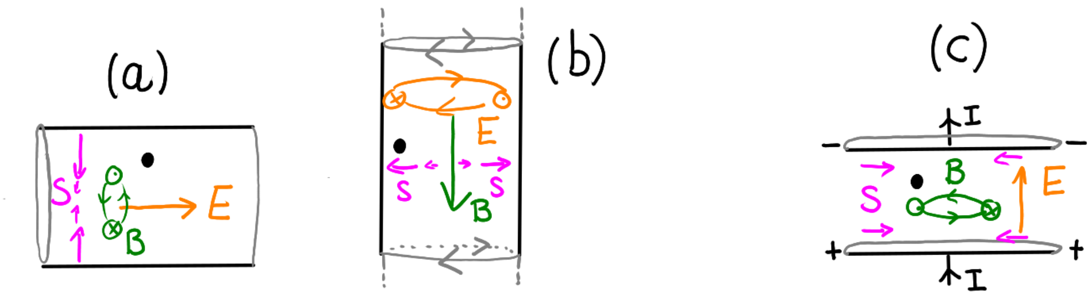
II. Pictures of fields [6p]

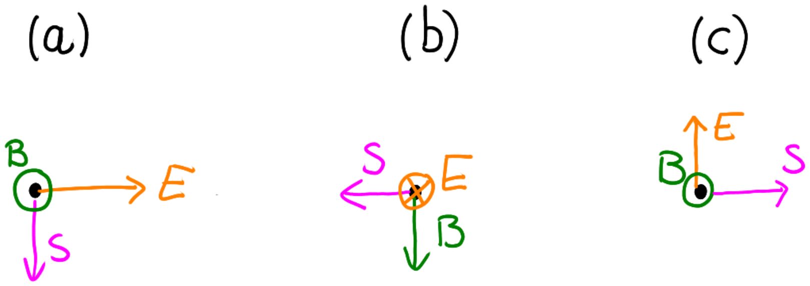


The three diagrams above show (a) a cylindrical ohmic wire with steady current flowing right, (b) a long solenoid with a *slowly decreasing* current directed clockwise when viewed from above, and (c) a broad parallel-plate capacitor that is charging slowly. The point P is always on a central cross-section of the object (in the dimension in/out of the paper), so it's inside the wire, solenoid, or capacitor, respectively.

5. (3 points) Draw or indicate the directions of the vectors **E**, **B**, and **S** at the point P in each problem. No explanations needed. (Don't just say 'x-direction', make clear what you mean. If a vector is zero, say so, don't omit it.)

Solution: Answers below. The first row sketches the field in the entire geometry, which helps answer the actual question in the second row.

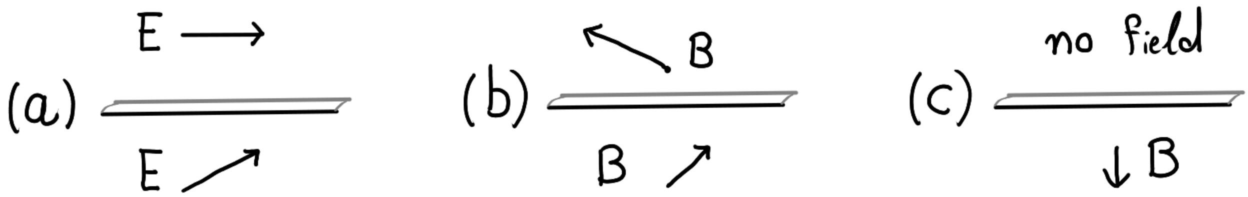




In (a), there is an electric field proportional to the current by Ohm's law. The current also causes a static magnetic field. In (b), we have our usual solenoid magnetic field. When the current decreases, a Faraday emf will try to keep it going, extracting energy from the magnetic field. In (c), a displacement current (or the current spreading over the plates, depending how you view it) due to the increasing electric field is present and causes a counterclockwise magnetic field.

Award one point for each correctly drawn triad (no "one-third" points). Ignore the magnitudes of the drawn vectors.

Three (completely different) diagrams below show situations very close to a surface, which is viewed edge-on. Other sources of fields may exist far from the surface. Only consider clearly visible differences, not minute imperfections of the drawing.



6. (3 points) For each diagram, state whether it is possible or not. For *possible* diagrams, *clearly* indicate the direction/sign of the surface current/charge. For *impossible* diagrams, state which boundary condition is violated (either give the formula or name the Maxwell equation it derives from). No explanations needed.

Solution:

- A is possible for a negative surface charge (field points more up on the bottom, for positive charge it would point more down)
- B is possible for a surface current pointing out of the page (right-hand rule, current adds a bit of field pointing left on the top and right on the bottom).
- C is impossible, because it violates the boundary condition derived from 'Noname's law' ($\nabla \cdot \mathbf{B} = 0$) / $B^{\perp}$above $-$ $B^{\perp}$below $= 0$.

Award one point for each fully correct answer. No half points for guessing possible/impossible.

III. Derivations [8 points]

7. (4 points) From the continuity equation for free charge, show that an initial free charge density $\rho_f(t = 0)$ in an ohmic medium dissipates with a characteristic time ϵ/σ, where ϵ is the permittivity and σ the conductivity of the medium.

Solution: (You saw this argument in the first problem of tutorial 25.)
The continuity equation for free charge is:

$$\frac{\partial \rho_f}{\partial t} = -\nabla \cdot \mathbf{J}_f. \quad [1\text{pt}]$$

Now use Ohm's law ($\mathbf{J}_f = \sigma \mathbf{E}$) and Gauss's law in matter ($\nabla \cdot \mathbf{E} = \rho_f/\epsilon$) to write:

$$\frac{\partial \rho_f}{\partial t} = -\sigma \nabla \cdot \mathbf{E} = -\frac{\sigma}{\epsilon}\rho_f \quad [2\text{pt}].$$

This is solved by $\rho_f(t) = \rho_f(0)e^{-t/(\epsilon/\sigma)}$, which e-folds with characteristic time $t = \frac{\epsilon}{\sigma}$ [1pt].
Forgive notational sloppiness around ϵ vs ϵ_0, ρ vs ρ_f, etc. Since the equations are linear, you may also set $\rho(0) = 1$ to simplify the algebra.

8. (4 points) Show that $u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$ satisfies the conservation of energy equation:

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{S} - \mathbf{E} \cdot \mathbf{J}, \quad (5)$$

with $\mathbf{J}$ the current density and $\mathbf{S}$ the Poynting vector. Hint: use a product rule from your formula sheet.

Solution: (Poynting's theorem: lecture 24, and Griffiths 8.1.2.)

Start by plugging in the expression for u on the LHS:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{2}\epsilon_0\frac{\partial E^2}{\partial t} + \frac{1}{2\mu_0}\frac{\partial B^2}{\partial t}\\ &= \epsilon_0 \mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} \quad \text{[1pt]}\end{aligned}$$

Now use Faraday's law and the Maxwell-Ampère law:

$$\frac{\partial u}{\partial t} = \mathbf{E}\cdot\left(\frac{1}{\mu_0}\nabla\times\mathbf{B} - \mathbf{J}\right) + \frac{1}{\mu_0}\mathbf{B}\cdot(-\nabla\times\mathbf{E}) \quad \text{[1pt]}$$

On the RHS, use $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ and the product rule #6 as hinted:

$$-\nabla\cdot\mathbf{S} = -\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}) = -\frac{1}{\mu_0}\left(\mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})\right) \quad \text{[1pt]}$$

The two equations above are equal up to $\mathbf{E}\cdot\mathbf{J}$, proving the result. [1pt].

IV. Levitations [4 points + 2 bonus]

A student suggests a lecture demonstration: levitate a tiny piece of aluminum foil with a laser pointer. Suppose the laser pointer has a power of $5.0\,\mathrm{mW}$ and an area of $1.0\,\mathrm{mm}^2$. The local gravitational acceleration is $g = 9.81\,\mathrm{m/s}^2$, and the density of aluminum is $2.71\,\mathrm{g/cm}^3$. Assume the foil is fully reflective and our piece has the same surface area as the laser beam.

9. (4 points) Compute the maximum thickness of a piece of aluminum foil that can be levitated by the radiation pressure of this laser. (Answer with a correct unit and number of significant figures.)

Solution: One point each for:

- The radiation force is $F = 2pA = 2IA/c$, with p radiation pressure, A the foil area, I radiation intensity, and c the speed of light. The factor 2 is because the light is reflected, not absorbed. You can earn half this point by just saying $p = 2I/c$ because the foil is fully reflective.
- The gravitational force is $F_g = mg = \rho V g = \rho D A g$ with D the thickness, m the mass and V the volume of the foil.
- Levitation requires $F_{\text{radiation}} = F_g$, so $D = \frac{2I}{\rho g c}$.
- Filling in the numbers and converting units, we get $D = 1.2526\times10^{-9}\,\mathrm{m} = 1.3\,\mathrm{nm}$.

A unit and 2 significant figures must be quoted to get the final point. If you are one digit off (so you answer Do not award the point for a substantially simplified final answer, such as "$D = 0.0\,\mathrm{m}$ because I don't know."

Remark: A $1.3\,\mathrm{nm}$ thick foil would actually be transparent to visible light $\lambda \approx 500\,\mathrm{nm}$, so this lecture demo isn't going to work. The moral is that radiation pressure is feeble for ordinary light-sources. If you are interested, look up *Crookes' radiometer*; now you understand how it does *not* work! Thermal physics is needed for the actual explanation.

A final bonus question follows. This is challenging and rewards only a few points. Attempt it only if you have sufficient time.

Another student proposes to add a partially reflective non-absorbing mirror with reflection coefficient $R = 0.999$ between the laser and the reflective foil. The mirror reflects light from either direction, and the provided coefficient characterizes the mirror as a whole (not one of its interfaces). Assume that the distance between the foil and the mirror is an exact multiple of the laser wavelength, that the foil and mirror are both perpendicular to the laser beam, and that light returned to the laser pointer is lost.

10. (2 bonus points) By what factor is the radiation pressure on the foil multiplied, compared to the setup without the mirror?

Solution:

- First, the incoming intensity is reduced by a factor $T = 1 - R$ because of the mirror between laser and foil.
- Because the distance is a multiple of the wavelength, light on different 'bounces' is in phase. Thus the electric fields of the waves add (their intensities don't).
- $R = (E_R/E_I)^2$, so the amplitude of the wave is reduced by $\sqrt{R}$ on each bounce.
- The total amplitude multiplier from all bounces is $\sum_{n=0}^{\infty} \sqrt{R}^n = \frac{1}{1-\sqrt{R}}$, and the intensity is multiplied by the square of this.
- Combined, we get a multiplier of $\frac{1-R}{(1-\sqrt{R})^2} = 3998$ ($\approx 4/(1-r)$)!

(So we'll only need a 5 µm foil now! That's better, but there are still some problems with the setup: for example, keeping the foil perfectly aligned with the laser beam and stably located at a multiple of a wavelength.)

Remark: this setup is a simplified version of a **Fabry-Pérot cavity**, which is used in many lasers and interferometers (such as the gravitational wave detectors LIGO/Virgo/KAGRA). The factor 3998 is the *buildup factor* of the cavity. You may study this in more detail in your Waves & Optics and other courses. For example, in reality both mirrors are partially reflective, the light may make some angle with respect to them, etc. Maxim recommends the thesis of Brian Harold May as a nice introduction/reference.

Scoring guide:

- A fully correct argument is worth 2 bonus points.
- Adding the intensity on each bounce (rather than the field) gives a multiplier of $\frac{1-R}{1-R} = 1$ (no enhancement); this is worth a maximum 1 bonus points. This is on average what would happen if the reflections were not in phase, for example because the distance was not chosen in a way related to the wavelength (if the cavity is not 'phase-locked').
- Forgetting the initial factor $T = 1 - R$ costs you half a point.
- Just saying that the intensity is multiplied by $T = 1-R$, and ignoring the bouncing completely is not worth any points.

This concludes the test. When you are finished, please:

- Write your **name and student number on every sheet**!
- If you used two sheets, mark them 'Sheet 1/2' and 'Sheet 2/2'. When you hand them in, bind them with **two paperclips** on opposite sides.
- Feed your solutions to the wooden box. Not in the box = not graded.
- Return your formula sheet and *unused* paper. Take this question paper, your cheat sheet, and *used* scratch paper home.