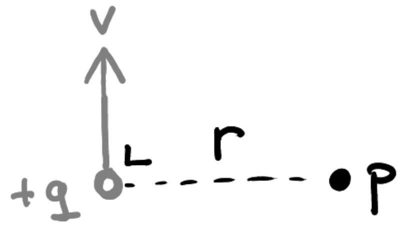
Electricity and Magnetism Test 6

17 June 2024, 8:30-10:30

You may use your double-sided A4 cheat sheet, the provided formula sheet, and a calculator. Good luck!

I. Short questions

1. (2 points) Explain in words what a gauge transformation is. No need to copy the defining equations.
2. (4 points) Define Lorenz and Coulomb gauge, and show what is required of ρ and $\mathbf{J}$ to satisfy both gauges simultaneously.
3. (5 points) A point charge $+q$ moves with speed v upwards, as shown in the figure. Using a field transformation, derive the $\mathbf{E}$ and $\mathbf{B}$ fields at the indicated point p (in the lab/paper frame), a distance r from the charge.
Clearly indicate the direction in which a field is positive, and carefully mention crucial steps in your reasoning.



4. **Choose and solve only ONE of 4A or 4B**. If it is unclear which one you picked, we will grade 4A.
   (a) (4 points) A student proposes the following two equations among tensors:

$$v^\mu = w^\nu S^\mu{}_\nu \tag{1}$$
$$a^\mu b_\mu = T^\mu{}_\nu \tag{2}$$

   For each equation, state how the **left**-hand side transforms under a change of coordinates, **then** explain whether the equation could be a law that has the same form for all inertial observers. (Consider a generic change of coordinates $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\mu$, without specifying $\Lambda^\mu{}_\nu$).

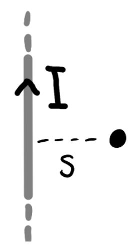
   (b) (4 points) In relativity, force density, or force per unit volume, is given by:

$$\mathbf{f} = \frac{d\mathbf{p}}{d\mathcal{V}dt}. \tag{3}$$

   Here $\mathbf{p}$ is the usual relativistic three-momentum, t denotes time, and $\mathcal{V}$ the volume of the object on which the force acts. All quantities are measured in the laboratory frame (*not* the object's rest frame). Show/argue that $\mathbf{f}$ is part of a four-vector f^μ, **and** give f^0.

II. Wire

Consider a very long (effectively infinite) neutral wire, and a point a distance s away from it, as drawn in the figure.



5. (2 points) Suppose the wire has a steady current I as drawn. Determine $\mathbf{B}$ at the point.

Suppose, instead, that the current suddenly turns on at a time t_0:

$$I(t) = \begin{cases} 0 & t < t_0 \\ I_0 & t > t_0 \end{cases} \tag{4}$$

6. (1 point) At what time do the electric and magnetic fields at the given point become non-zero? Call this time t_s.
7. (5 points) Determine the directions of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{E}$, for $t > t_s$, at the given point. (You do *not* need to compute the magnitudes.)

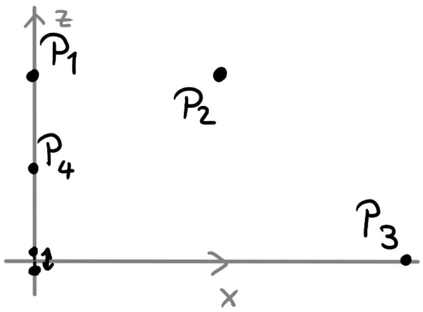
III. Radio

Two equal and opposite charges dance around the origin as shown, so that the separation between the charges is $d \cos \omega t$ for some positive ω. We observe the fields a distance r from the origin.

8. (3 points) Suppose $d \ll r \ll \frac{c}{\omega}$. Explain to what power of r the electric field is proportional. If you use a standard result, explain why it applies, do not derive it.

Now let $d \ll \frac{c}{\omega} \ll r$, and suppose the charges oscillate in the z-direction. Consider four points:

$$P_1 = (0, 0, \ell) \qquad P_2 = (\ell, 0, \ell)$$
$$P_3 = (2\ell, 0, 0) \qquad P_4 = (0, 0, \ell/2)$$



as shown in the figure. Here ℓ is some positive distance $\gg \frac{c}{\omega}$.

9. (5 points) Rank, from small to large, the intensity $I = \langle S \rangle$ of radiation at the points. Explain your answer.

For example, answer $I_2 < I_3 = I_4 < I_1$ if you think the intensity is largest at 1, weakest at 2, and in between and equal at 3 and 4. You do not need to compute the intensities in full, just explain your reasoning and calculations you did to determine the rank order.

IV. Light on the move

Consider a monochromatic electromagnetic plane wave with the following electric field:

$$\mathbf{E} = E_0 \cos[\phi(x, t)]\hat{\mathbf{z}} \tag{5}$$

where $\phi(x, t) = kx - \omega t$ is the phase of the wave, and ω and k are positive constants so that $\omega = ck$.

An observer traveling with velocity v in the x-direction must agree with us on the phase of the wave at any spacetime event. However, they will express the phase as $\bar{\phi}(\bar{x}, \bar{t}) = \bar{k}\bar{x} - \bar{\omega}\bar{t}$ in their coordinates, where $\bar{k}$ and $\bar{\omega}$ are different constants.

10. (3 points) Show that $\bar{k} = \gamma(k - \omega v/c^2)$ and $\bar{\omega} = \gamma(\omega - kv)$.

11. (5 points) Suppose the observer is moving at $\beta = 3/5$ and measures the wave's **speed** $\bar{v}$, **wavelength** $\bar{\lambda}$, and the electric field's **amplitude** $\bar{E}_0$. Express each in the same quantity measured in our original frame (v, λ, E_0). Hint: there is also a magnetic field…

When you are finished:

- Clearly strikethrough or otherwise mark or mar what we should *not* grade.
- Write your name and student number on your solutions – both sheets, if you used two.
- If you used two sheets, please mark them 'Sheet 1/2' and 'Sheet 2/2'.
- Place solutions in the box with your student number.
- Add a mark to the right of your name on the list in that box – under '✓ here!'
- Return the formula sheet and *unused* paper to their corresponding stacks.
- Take the piece of paper you are currently reading, and any *used* scratch paper, home. Double check there is nothing you want handed in on these.
- Exit the hall in the back, not where you entered.
- Have a great summer!