

Electricity and Magnetism Test 6

17 June 2024, 8:30-10:30

You may use your double-sided A4 cheat sheet, the provided formula sheet, and a calculator. Good luck!

Solution: Because question 4a contained a typo (and taking into consideration that question 11 redefined a symbol), we shifted the entire question I.4 to a bonus question. This applies regardless of whether you did 4a or 4b. It simply means the score needed to get a 10 is four points less. If you handed in test 6, your grade will be:

$$T6 = 1 + 9 \cdot \frac{\text{Total points}}{35} + \text{Tutorial bonus (if applicable)}.$$

Instructions for grading TAs:

- If several students make a similar mistake, please write that down AND agree on a consistent way to score it!
- Only award full integer points equal or larger than zero. Award fractional points only for specifically described circumstances below. If you need a new case for 0.5pt, agree with other TAs on a rule and write that down.
- If the question asks for an explanation, calculation, determination, argumentation, etc., award no points if this is missing or clearly incorrect / incoherent.
- Subtract points for each mistake only once, unless the error substantially simplifies or alters the rest of the problem.
- Pay close attention to answers for ‘show that ...’ questions. Making two mistakes that miraculously cancel each other should be awarded *fewer* points than making one mistake and not reaching the result.

I. Short questions

1. (2 points) Explain in words what a gauge transformation is. No need to copy the defining equations.

Solution:

- A gauge transformation changes the scalar and vector potentials (V and $\mathbf{A}$) .../ transforms between two pairs $(V', \mathbf{A}')$ and $(V, \mathbf{A})$
- in a way that does not change the electric and magnetic fields $\mathbf{B}$, $\mathbf{E}$ / any predictions of the theory / the solution to Maxwell’s equations.

‘transformations of $\mathbf{A}$ and V that do not change (the form of) Maxwell’s equations’: only the first point. This is not sufficiently precise. A transformation that maps one solution of Maxwell’s equations to another, like a rotation in space, is also said to leave (the form of) the equations unchanged.

‘transformations that do not change $\mathbf{E}$ and $\mathbf{B}$ ’: only the second point, you have to say what is being transformed.

Many students mentioned that the transformations make ‘The Math’ easier. That can of course be said for many transformations. I wouldn’t recommend transformations that make The Math more difficult.

2. (4 points) Define Lorenz and Coulomb gauge, and show what is required of ρ and $\mathbf{J}$ to satisfy both gauges simultaneously.

Solution: One point each for:

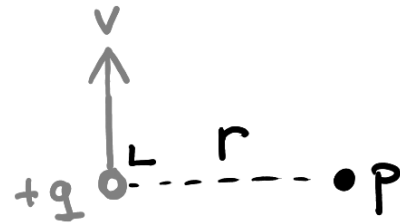
- Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ and Lorenz gauge: $\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t}$ (or $-\epsilon_0 \mu_0 \frac{\partial V}{\partial t}$). Both must be correct to get the point (no half points).
- These are equivalent if V is time-independent
- which requires ρ to be time-independent ($\nabla^2 V = -\frac{\rho}{\epsilon_0}$ in Coulomb gauge, same in Lorenz gauge if V is time-independent).
- From the continuity equation, this means $\mathbf{J}$ must be free of divergence.

For the fourth point, if $\nabla \cdot \mathbf{J} = 0$ is derived, but this is called 'steady current', subtract 0.5 point. A divergence-free current is not necessarily time-independent; for example, you can consider a loop of alternating current.

Common mistake: saying that V (or ρ) must be zero, or a constant, because its time derivative is zero.

3. (5 points) A point charge $+q$ moves with speed v upwards, as shown in the figure. Using a field transformation, derive the $\mathbf{E}$ and $\mathbf{B}$ fields at the indicated point p (in the lab/paper frame), a distance r from the charge.

Clearly indicate the direction in which a field is positive, and carefully mention crucial steps in your reasoning.



Solution:

- State fields in the rest frame of the charge $\bar{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \frac{q}{\bar{r}^2}$, $\bar{\mathbf{B}} = 0$ [1pt]
- Distance is the same in both frames / no Lorentz contraction perpendicular of motion / $r = \bar{r}$ – [one of these must be mentioned]. [1pt]
- Find $\mathbf{E}$ with a field transformation. Because p is at a right angle with respect to the moving charge, $E = \gamma \bar{E}$ in the same direction. Thus

$$E = \frac{\gamma}{4\pi\epsilon_0} \frac{q}{r^2} \quad [1\text{pt}]$$

- There is no magnetic field in the rest frame, so in the moving frame $\mathbf{B} = \frac{\mathbf{v} \times \mathbf{E}}{c^2}$,

$$B = \frac{\gamma v}{4\pi\epsilon_0 c^2} \frac{q}{r^2} = \frac{\gamma v \mu_0}{4\pi} \frac{q}{r^2}$$

[1pt, final simplification involving μ_0 is optional]

- Correct directions for both fields: At p , $\mathbf{E}$ points to the right away from the charge, $\mathbf{B}$ points down into the paper. [1pt]

Denoting the lab frame with bars, and the rest frame without bars, is of course also fine.

A few students did not notice that p was at right angles and worked on the full transformation law for arbitrary points! As long as the directions are drawn correctly for the given p , you can

still get the full points. If p is moved to a new location in the student's drawing, but directions are drawn correctly for that point, award 4 points for an otherwise completely correct solution (which will contain an unknown angle θ).

Quoting the field of a moving point charge from memory/cheat sheet, then filling in $\theta = \pi/2$: substantial simplification (the question told you to use a field transformation). Max 2 points: 1 for the realization this formula applies with $\theta = \pi/2$, and 1 for drawing the correct directions.

Common mistakes: using a wrong coordinate system or mixing Cartesian and spherical in a wrong way.

4. **Choose and solve only ONE of 4A or 4B.** If it is unclear which one you picked, we will grade 4A.

(a) (4 points) A student proposes the following two equations among tensors:

$$v^\mu = w^\nu S^\mu{}_\nu \quad (1)$$

$$a^\mu b_\mu = T^\mu{}_\nu \quad (2)$$

For each equation, state how the **left**-hand side transforms under a change of coordinates, **then** explain whether the equation could be a law that has the same form for all inertial observers. (Consider a generic change of coordinates $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$, without specifying $\Lambda^\mu{}_\nu$).

Solution: Eq. 1: left-hand side transforms as (a contravariant) four-vector $\bar{v}^\mu = \Lambda^\mu{}_\nu v^\nu$ [1pt]. [Also allow $\bar{v}^\mu = \Lambda^\mu{}_\nu v^\mu$, which makes no sense, but matches the unfortunate typo in the hint...]

Eq. 2: left-hand side transforms as a scalar / is invariant / $\bar{a}^\mu \bar{a}_\mu = a^\mu a_\mu$. [1pt]

Equation 1 can be a law, the right-hand side also transforms as a (contravariant) four-vector (one free μ index) / transforms the same way as the left-hand side. [1pt]

Equation 2 cannot be a law, the right-hand side transforms differently / is not invariant / transforms as a (1,1) tensor rather than a scalar. [1pt]

Common mistake: saying the second equation's left-hand side transforms with two Lorentz factors.

(b) (4 points) In relativity, force density, or force per unit volume, is given by:

$$\mathbf{f} = \frac{d\mathbf{p}}{d\mathcal{V}dt}. \quad (3)$$

Here $\mathbf{p}$ is the usual relativistic three-momentum, t denotes time, and $\mathcal{V}$ the volume of the object on which the force acts. All quantities are measured in the laboratory frame (*not* the object's rest frame). Show/argue that $\mathbf{f}$ is part of a four-vector f^μ , **and** give f^0 .

Solution: Under Lorentz boosts, the volume contracts [1pt] while time dilates, so the denominator is invariant [1pt]. (In particular, $d\mathcal{V}dt = (\mathcal{V}_0/\gamma)(\gamma d\tau) = d\mathcal{V}_0 d\tau$, with $\mathcal{V}_0$ and τ the proper volume and time for the object.)

But $\mathbf{p}$ are spatial components of the four-momentum p^μ . Thus force density is part of a four vector [1pt] with $f^0 = p^0/(d\mathcal{V}dt)$ [1pt] ($= \frac{1}{c} \frac{dE}{d\mathcal{V}dt}$, power delivered per unit volume, divided by c).

Do not award the last point simply for writing $p^0/(d\mathcal{V}dt)$, without saying what is meant by p^μ (the four-momentum).

" $f^\mu = p^\mu/(d\mathcal{V}dt)$, with p^μ the four-momentum, is a four-vector that has $\mathbf{f}$ as its spatial components, so $f^0 = p^0/(d\mathcal{V}dt)$ ": this is worth two points. Mentioning that the denominator is invariant gets another point. Explaining why it is invariant gives the final point.

Common mistake: ignoring the denominator and focusing only on $\mathbf{p}$.

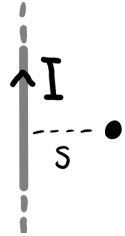
II. Wire

Consider a very long (effectively infinite) neutral wire, and a point a distance s away from it, as drawn in the figure.

5. (2 points) Suppose the wire has a steady current I as drawn. Determine $\mathbf{B}$ at the point.

Solution: The current is axial so $\mathbf{B}$ must be circumferential. Use Ampere's law, Amperian loop around the wire (drawing is best): $2\pi s B = \mu_0 I_{\text{encl}}$ so $\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$. (If we take $I \propto \hat{\mathbf{z}}$)

Award only one point if there is a single error, e.g. missing direction or a deficient explanation (e.g. neither mentioning nor drawing what loop was used). No points for stating the correct result without any derivation.



Suppose, instead, that the current suddenly turns on at a time t_0 :

$$I(t) = \begin{cases} 0 & t < t_0 \\ I_0 & t > t_0 \end{cases} \quad (4)$$

6. (1 point) At what time do the electric and magnetic fields at the given point become non-zero? Call this time t_s .

Solution: $t_s = t_0 + r/c$, the time needed for a light signal from the wire to reach s . ($t_r = t - r/c > t_0 \implies t > t_0 + r/c$)

7. (5 points) Determine the directions of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{E}$, for $t > t_s$, at the given point. (You do *not* need to compute the magnitudes.)

Solution:

- $\mathbf{A} \propto \int \frac{\mathbf{J}(\dots)}{r} d\tau'$, so $\mathbf{A}$ is in the same direction as the current ($\hat{\mathbf{z}}$). [1pt]
- $\mathbf{B} = \nabla \times \mathbf{A}$, so it is circumferential and into the paper. [1pt] Or: as $t \rightarrow \infty$, we should approach the statics solution, so $\mathbf{B}$ is circumferential into the paper.
- $V = 0$ since $\rho = 0$ (Lorenz gauge) [1pt]
- $\mathbf{A}$ increases with time as contributions from more of the wire start to matter (have $t_r > t_0$) [1pt]
- Thus $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{A}}{\partial t}$ is in the direction of $-\hat{\mathbf{z}}$, opposite to the current. [1pt]

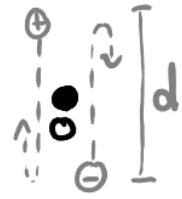
The last three points (for $\mathbf{E}$'s direction) can also be awarded by arguing $\mathbf{S}$ must point away from the wire (turning on the current requires accelerating charges, emitting radiation), then using $\mathbf{S}_{\mu_0} = \mathbf{E} \times \mathbf{B}$ and the right-hand rule.

Stating or assuming $V = 0$ without any explanation: subtract the third point.

III. Radio

Two equal and opposite charges dance around the origin as shown, so that the separation between the charges is $d \cos \omega t$ for some positive ω . We observe the fields a distance r from the origin.

8. (3 points) Suppose $d \ll r \ll \frac{c}{\omega}$. Explain to what power of r the electric field is proportional. If you use a standard result, explain why it applies, do not derive it.



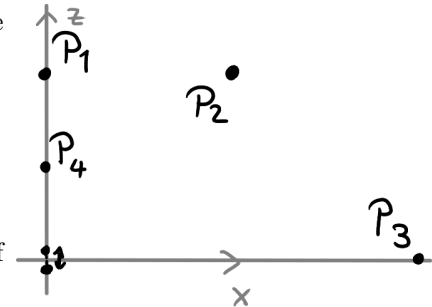
Solution: Since $d \ll r$, we can still treat the system as a dipole (ignore higher-order multipole moments) [1pt]. Because ω is extremely small, we are in (quasi)statics [1pt], so we expect a $1/r^3$ field as for a dipole in electrostatics [1pt].

Now let $d \ll \frac{c}{\omega} \ll r$, and suppose the charges oscillate in the z-direction. Consider four points:

$$\begin{aligned} P_1 &= (0, 0, \ell) & P_2 &= (\ell, 0, \ell) \\ P_3 &= (2\ell, 0, 0) & P_4 &= (0, 0, \ell/2) \end{aligned}$$

as shown in the figure. Here ℓ is some positive distance $\gg \frac{c}{\omega}$.

9. (5 points) Rank, from small to large, the intensity $I = \langle S \rangle$ of radiation at the points. Explain your answer.



For example, answer $I_2 < I_3 = I_4 < I_1$ if you think the intensity is largest at 1, weakest at 2, and in between and equal at 3 and 4. You do not need to compute the intensities in full, just explain your reasoning and calculations you did to determine the rank order.

Solution: Answer is $I_1 = I_4 < I_2 = I_3$ (or equivalent).

- Dipoles do not radiate in the direction of oscillation (or use the math, $\sin 0 = \sin \pi/2 = 0$). [1 pt]
- Thus intensity is lowest (zero) at both I_1 and I_4 . Ranking starts with $I_1 = I_4 < \dots$ [1pt]
- For dipole radiation, $I \propto \frac{\sin^2 \theta}{r^2}$ (either directly from cheat sheet, or use $I \propto \mathbf{S} \propto \mathbf{E} \times \mathbf{B} \propto E^2$, as $B \propto E$) [1pt]
- $(r_2, \theta_2) = (\sqrt{2}\ell, \pi/4)$ while $(r_3, \theta_3) = (2\ell, \pi/2)$. [1pt]
- Correct conclusion: $I_2 = I_3$. Ranking ends with $\dots < I_2 = I_3$ [1pt]

IV. Light on the move

Consider a monochromatic electromagnetic plane wave with the following electric field:

$$\mathbf{E} = E_0 \cos[\phi(x, t)] \hat{\mathbf{z}} \quad (5)$$

where $\phi(x, t) = kx - \omega t$ is the phase of the wave, and ω and k are positive constants so that $\omega = ck$.

An observer traveling with velocity v in the x-direction must agree with us on the phase of the wave at any spacetime event. However, they will express the phase as $\bar{\phi}(\bar{x}, \bar{t}) = \bar{k}\bar{x} - \bar{\omega}\bar{t}$ in their coordinates, where $\bar{k}$ and $\bar{\omega}$ are different constants.

10. (3 points) Show that $\bar{k} = \gamma(k - \omega v/c^2)$ and $\bar{\omega} = \gamma(\omega - kv)$.

Solution: This can be solved using the inverse Lorentz transforms or the ordinary Lorentz transform. The first is much easier, but the latter also works.

Inverse Lorentz transformation:

$$\begin{aligned}x &= \gamma(\bar{x} + v\bar{t}) \\ t &= \gamma(\bar{t} + v\bar{x}/c^2)\end{aligned}$$

to get

$$\begin{aligned}\phi(x, t) &= \bar{\phi}(\bar{t}, \bar{x}) \\ \omega\gamma(\bar{t} + v\bar{x}/c^2) - k\gamma(\bar{x} + v\bar{t}) &= \bar{\omega}\bar{t} - \bar{k}\bar{x}\end{aligned}$$

[1pt]. This has to be equal for all $\bar{x}$ and $\bar{t}$, so we can collect their coefficients and equate them separately [1pt]:

$$\begin{aligned}\omega\gamma\bar{t} + \omega\gamma v\bar{x}/c^2 - k\gamma\bar{x} - k\gamma v\bar{t} &= \bar{\omega}\bar{t} - \bar{k}\bar{x} \\ (\omega\gamma - k\gamma v)\bar{t} + (\omega\gamma v/c^2 - k\gamma)\bar{x} &= \bar{\omega}\bar{t} - \bar{k}\bar{x}\end{aligned}$$

implying the result. [1pt]

Ordinary Lorentz transform:

$$\begin{aligned}\bar{x} &= \gamma(x - vt) \\ \bar{t} &= \gamma(t - vx/c^2)\end{aligned}$$

Gets us

$$\begin{aligned}\phi(x, t) &= \bar{\phi}(\bar{t}, \bar{x}) \\ \omega t - kx &= \bar{\omega}\bar{t} - \bar{k}\bar{x} \\ &= \bar{\omega}\gamma(t - vx/c^2) - \bar{k}\gamma(x - vt)\end{aligned}$$

[1pt] This must hold for all t and all x , so we equate the t and x coefficients separately:

$$\begin{aligned}\omega &= \gamma\bar{\omega} + \gamma v\bar{k} \\ -k &= -\bar{\omega}\gamma v/c^2 - \bar{k}\gamma\end{aligned}$$

[1pt]. From here there are several routes, each worth the final [1pt] if done correctly. You could solve the system by substitution or elimination. You can also note that these are two independent linear equations with two unknowns, which has a unique solution (if any), so it suffices to check the given solution (which still takes a bit of algebra).

Award 0.5pt instead of the final 1pt if a student correctly derives one equation but goofs in the other.

11. (5 points) Suppose the observer is moving at $\beta = 3/5$ and measures the wave's **speed** $\bar{v}$, **wavelength** $\bar{\lambda}$, and the electric field's **amplitude** $\bar{E}_0$. Express each in the same quantity measured in our original frame (v , λ , E_0). Hint: there is also a magnetic field...

Solution: Velocity of the wave is equal, $\bar{c} = c$. This is an electromagnetic wave and the speed of light is invariant / because ω/k is the phase velocity and (from the results above) we see this is the same in both frames. [1pt]

[If a student confuses v for the speed of the observer, because the previous subquestion used it

in that sense, fine – the conclusion is still the same, the relative speed $\bar{v} = v = \beta c$ also does not change and the relative If a student confuses v for the speed of the observer, because the previous question used it in that sense, no harm done – the conclusion is still the same, the relative speed $\bar{v} = v = \beta c$ does not change and the relative velocity flips sign. (Also allow $\bar{v} = -v$.)]

$\bar{k} = \gamma(k - \omega v/c^2) = \gamma(1 - \beta)k = k/2$. (where we used $\omega = ck$) [1pt]. Wavelength is $\propto 1/k$, so $\bar{\lambda} = 2\lambda$. [1pt]

This is an electromagnetic plane wave, so the magnetic field is $\mathbf{B} = \frac{\hat{\mathbf{k}} \times \mathbf{E}}{c} = -E\hat{\mathbf{y}}/c$ [1pt].

The amplitude of $\mathbf{E}$ is reduced, as we can see from the field transformation

$$\begin{aligned}(\bar{E}_x &= E_x = 0) \\ (\bar{E}_y &= \gamma(E_y - vB_z) = 0) \\ \bar{E}_z &= \gamma(E_z + vB_y) = \gamma(1 - \beta)E = E/2\end{aligned}$$

So the amplitude $\bar{E}_0 = E_0/2$. [1pt]

Ignoring the magnetic field, and concluding $\bar{E} = \gamma E = 5E/4$: substantial simplification, give 0.5 pt instead of the final two.

When you are finished:

- Clearly strikethrough or otherwise mark or mar what we should *not* grade.
- Write your name and student number on your solutions – both sheets, if you used two.
- If you used two sheets, please mark them ‘Sheet 1/2’ and ‘Sheet 2/2’.
- Place solutions in the box with your student number.
- Add a mark to the right of your name on the list in that box – under ‘✓ here!’
- Return the formula sheet and *unused* paper to their corresponding stacks.
- Take the piece of paper you are currently reading, and any *used* scratch paper, home. Double check there is nothing you want handed in on these.
- Exit the hall in the back, not where you entered.
- Have a great summer!