Electricity and Magnetism Test 6

16 June 2025, 8:30 – 10:30

- You may use your double-sided A4 cheat sheet, the provided formula sheet, and a calculator.
- Please leave margins for grading. Do *not* use this paper, or the white scratch paper, for final answers.
- Clearly indicate directions of vector quantities. Make or copy diagrams if it helps you.
- The maximum score is 32 points. Good luck!

Solution:
If you handed in the test, your grade will be:

$$\min\left(\ 10,\ \ 1 + 9\cdot\frac{\text{Total points}}{32} + \text{Tutorial bonus (if applicable)}\right).$$

Instructions for grading TAs:

- If several students make a similar mistake, please write that down AND agree on a consistent way to score it!
- Only award full integer points equal or larger than zero. Exception: sometimes the rubric says half point may be awarded for a specific circumstance. If you think we need a new case for 0.5pt, agree with other TAs on a rule and write that down (so we can update the rubric).
- If the question asks for an explanation, calculation, determination, argumentation, etc., award no points if this is missing or clearly incorrect / incoherent.
- Subtract points for each mistake only once, unless the error substantially simplifies or alters the rest of the problem.
- Pay close attention to answers for 'show that ...' questions. Making two mistakes that miraculously cancel each other should be awarded *fewer* points than making one mistake and not reaching the result.

I. Short questions [13p]

1. Consider two events with $\Delta x^\mu \Delta x_\mu > 0$ (under the sign conventions used throughout this course).
   (a) (1 point) What does $\Delta x^\mu \Delta x_\mu$ represent in general?

   Solution: It is the (invariant) **spacetime interval** (squared) between two events. [1pt]
   To see this, restore the sum and use our index notation conventions:

   $$\begin{aligned}\Delta x^\mu \Delta x_\mu &= \sum_\mu \Delta x^\mu \Delta x_\mu \\ &= \Delta x^0 \Delta x_0 + \Delta x^1 \Delta x_1 + \Delta x^2 \Delta x_2 + \Delta x^3 \Delta x_3 \\ &= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\end{aligned}$$

   (b) (2 points) $\sqrt{\Delta x^\mu \Delta x_\mu}$ for these two events is a specific physical quantity measured by a specific observer. Give the name of this quantity, and specify what observer it is measured by.

   Solution: It is the **proper distance** between the two events [1pt], the distance between the events in the frame where they are simultaneous. [1pt]
   One point for: The proper time between the events, the time between events according to an observer passing through both. (This answer uses the wrong metric sign convention.)

2. A charged particle is moving in a circular orbit with constant radius and constant speed.

(a) (2 points) Briefly explain why an external source of energy is needed to keep the particle in this orbit. You can quote results derived in class/Griffiths without derivation.

> **Solution:**
>
> - When moving in a circular orbit with constant radius, the particle accelerates centripetally ($a = v^2/R$). [1pt]
> - Any accelerating charged particle radiates (Larmor formula), so this particle would lose energy (and spiral inwards) unless some external source compensates. [1pt]

(b) (1 point) Suppose the circle is large enough that the center can be considered far from the particle for the purpose of calculating the fields. Can we detect electromagnetic waves at the point at the center of the circular orbit? (Explain very briefly, one sentence suffices.)

> **Solution:** No, because no radiation is emitted along the direction of acceleration. (The angle $\theta = 0$ with respect to $\ddot{\mathbf{p}}$.)
> "No, because this would violate rotational symmetry / radiation from all sides cancels out": incorrect. The *time-averaged* radiation flux measured at the center has to be zero by symmetry, but at any one instant the problem has no rotational symmetry. The charge is at some specific point in the orbit.

3. (3 points) You measure a magnetic field strength of 1.0 Tesla at some point P in space. Your friend moving at $v = 0.60c$ relative to you measures no *electric* field at P. Find the minimum and maximum possible magnitude in V/m of the electric field that you measure at P.

> **Solution:** Call your friend's frame S and your frame $\bar{S}$. Since your friend sees no electric field, the fields in your frame (or another inertial frame) obey $\bar{\mathbf{E}} = \mathbf{v} \times \bar{\mathbf{B}}$, with $\mathbf{v}$ the frame velocity relative to S [1pt]. (If you do not remember this special case, you can also derive it from the field transformation rules.)
>
> If $\mathbf{v}$ is orthogonal to $\bar{\mathbf{B}}$, the electric field you measure has its maximum value $vB = 0.6c \cdot 1T = 0.6 \cdot 3 \times 10^8\,\mathrm{m/s} \cdot 1\,\mathrm{T} = 1.8 \times 10^8\,\mathrm{V/m}$ [1pt].
>
> *Scoring note: Getting to $E = 0.6c \cdot 1T$ and stopping there (e.g. because you don't know how to convert the units) is worth half a point. Saying $E = 0.6c$ is not worth a point, then the units make no sense.*
>
> If $\mathbf{v}$ and $\mathbf{B}$ are parallel, the electric field you measure is zero [1pt].

4. (4 points) Show that the Maxwell-Ampere law $\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ becomes $\mu_0 \mathbf{J} = -\Box^2 \mathbf{A}$ in Lorenz gauge.

> **Solution:** Substitute the expressions of the fields in terms of the potentials:
>
> $$\mathbf{B} = \nabla \times \mathbf{A},$$
> $$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t},$$
>
> into the Maxwell-Ampere law as given, and use second derivative rule #11
>
> $$\begin{aligned} \mu_0 \mathbf{J} &= \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\ &= \nabla \times (\nabla \times \mathbf{A}) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \frac{1}{c^2}\left(\nabla \frac{\partial V}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \\ &= \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}) - (\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\mathbf{A}. \end{aligned}$$

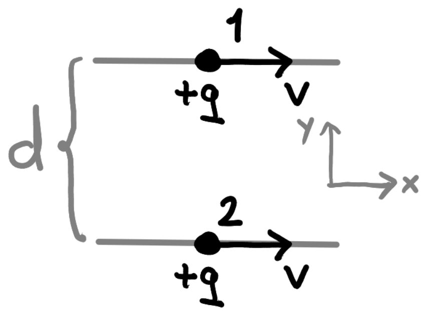
The first term in parentheses is zero by the Lorenz gauge condition ($\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$), and the second term in parentheses is the d'Alembertian $\Box^2$, which gives the required result.

One point each for:

- Correctly recalling the Lorenz gauge condition (listing it is sufficient for this point)
- Substitute the potential-to-field expressions into the MxA-law (just listing the expressions does *not* give this point, they are on the formula sheet)
- Correctly using the second derivative rule #11
- Correct and understandable rest of the derivation.

II. Two point charges [7pt]

Two point charges, each of charge $+q$, move in parallel with speed v (on straight rails) in the $+\hat{\mathbf{x}}$ direction. The rails are a distance d apart in the $+\hat{\mathbf{y}}$ direction, with charge number 1 on top. See the figure on the right for illustration. (Do *not* assume $v \ll c$ in this question!)



5. (1 point) How far apart are the charges in the frame where they are at rest?

Solution: Still d (there is no length contraction in directions perpendicular to the motion). [1pt]

6. (6 points) Find the force on charge 1 due to the fields of charge 2, in the frame specified and drawn in the question. Start from the frame where the charges are at rest, then use the field transformation rules.

Solution: *Adapted from test 6 2021 / problem 12.46 Griffiths 4th edition.*

Consider the frame S where the charges are at rest. The electric field of a point charge is $\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$, so $\mathbf{E} = +\frac{q}{4\pi\epsilon_0 d^2}\hat{\mathbf{y}}$ at charge 1. [1pt]

Scoring note: the point is not for writing down the field of a point charge, but for inserting $r = d$ and getting the correct direction. Also award this point if a spurious γ is added but this is consistent with an incorrect answer to the previous question.

There is no magnetic field in the rest frame. Transform to the lab frame $\bar{S}$, so with velocity $-v$ in the x direction. Using the field transformation rules, we find:

$$\bar{E}_y = \gamma E_y \quad [1\text{pt}]$$
$$\bar{B}_z = +\gamma \frac{v}{c^2} E_y \qquad [1\text{pt}]$$

and other components are zero since they do not involve E_y.

Scoring note: Students may assign $S/\bar{S}$ differently. Award these two points only if it is clear what frame is barred (explicitly written, or implicit from a clear continuation of the solution). Without this there is no way to check the sign / powers of gamma.

Finally, for the force on the charge, use the Lorenz force law:

$$\begin{aligned}
\bar{\mathbf{F}} &= q\left(\bar{\mathbf{E}} + \mathbf{v} \times \bar{\mathbf{B}}\right) \\
&= q\left(\gamma E_y \hat{\mathbf{y}} + \left(v \cdot \gamma \frac{v}{c^2} E_y\right)(\mathbf{x} \times \hat{\mathbf{z}})\right) \; [1\text{pt}] \\
&= q E_y \gamma (1 - v^2/c^2)\hat{\mathbf{y}} \;\; [1\text{pt}] \\
&= \frac{q E_y}{\gamma}\hat{\mathbf{y}}
\end{aligned}$$

(which could be more simply derived from the force transformation laws, but the question tells you to use field transformations). The final result is

$$\bar{\mathbf{F}} = \frac{q^2}{4\pi\epsilon_0 d^2 \gamma}\hat{\mathbf{y}} = \frac{q^2}{4\pi\epsilon_0 d^2}\sqrt{1-\frac{v^2}{c^2}}\hat{\mathbf{y}}. \quad \text{[1pt]} \tag{1}$$

(Either form is acceptable.)

Scoring note: One point for any solution that fills in the lab-frame fields in the Lorentz force law (even if vector directions are ignored). Second point for a correct solution with a calculation mistake, e.g. sign mistake in the cross product. Third point for a fully consistent result given the fields found earlier.

Mistakes of note A few students forgot the electric field of a stationary point charge. (You can derive this from Gauss' law, which is on your formula sheet...)

III. Rotated dipole [6p]

You are at a distance $41\pi c/\omega$ in the $+\hat{\mathbf{y}}$ direction from a tiny, forever-oscillating dipole with $\mathbf{p}(t) = p_0 \cos(\omega t)\hat{\mathbf{x}}$ with $p_0 > 0$. The 'radiation zone' is the region of space where the usual dipole radiation solution we derived in class is accurate.

7. (1 point) Briefly justify why you are in the radiation zone. [1p]

Solution: To be in the radiation zone, we need $\lambda = 2\pi c/\omega \ll r$. (And $d \ll \lambda$, but the question specifies that the dipole is 'tiny'.) Since $r = 41\pi c/\omega$, and $2 \ll 41$, we're good.

8. (5 points) Determine the *direction* in Cartesian coordinates of the electric field at $t = 0$ at your position. If you rotate or change coordinates, make clear how you do this (perhaps with a drawing).

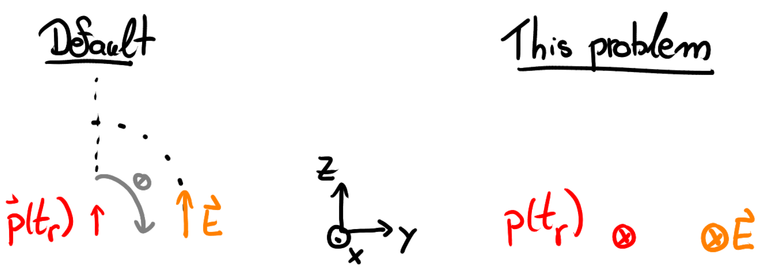
Solution:

- The dipole is at retarded time $t - r/c = 0 - 41\pi/\omega$ [1pt]
- So $\mathbf{p}(t_r) = p_0 \cos(-41\pi)\,\hat{\mathbf{x}} = -p_0\hat{\mathbf{x}}$ [1pt]

Now continue in one of two ways.

Method 1: rotating the standard solution

- We observe at an angle $\theta = \pi/2$ to $\mathbf{p}(t_r)$. In the standard setup (class/Griffiths), $\mathbf{p}(t_r) \propto +\hat{\mathbf{z}}$, and $\mathbf{E} \propto -\hat{\theta}(\pi/2) = +\hat{\mathbf{z}}$ [1pt].
- So we must have $\mathbf{E} \propto \mathbf{p}(t_r)$ in this problem (we just chose coordinates differently). Alternatively, choosing coordinates and drawing (see below), we see that the problem is rotated 90 degrees into the page compared to the standard setup. [1pt]
- We conclude $\mathbf{E} \propto -\hat{\mathbf{x}}$. [1pt]



Method 2: analytic formula

- In general, $\mathbf{E} \propto \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(\mathbf{t_r}))$ (from Griffiths/tutorials). In this problem, $\hat{\mathbf{r}} = \hat{\mathbf{y}}$. [1pt]
- Moreover, $\ddot{\mathbf{p}}(\mathbf{t_r}) \propto -\mathbf{p} \propto +\hat{\mathbf{x}}$. [1pt]
- Thus $\mathbf{E} \propto \hat{\mathbf{y}} \times (\hat{\mathbf{y}} \times \hat{\mathbf{x}}) = \hat{\mathbf{y}} \times -\hat{\mathbf{z}} = -\hat{\mathbf{x}}$ [1pt].

Scoring note: A correct solution for the standard dipole, but with incorrect (or missing, or completely unclear, or apparently guessed) rotation to the current frame is worth a maximum of 3 points. Award four points for e.g. a rotation correct up to a minus sign error.

Common mistakes Several students did not know how to find the direction of the field at all and guessed instead. It is also very easy to get one or more minus sign errors here: from $\cos -41\pi = -1$ (from retardation), or from $\hat{\theta}(\pi/2) = -\hat{\mathbf{z}}$ (from geometry), or from $\ddot{\mathbf{p}} = -\mathbf{p}$ (from harmonic oscillation). If you miss two signs, you accidentally get the right answer, but it will cost you points twice!

IV. Can you gauge this? [6p]

Given some vector potential $\mathbf{A}(\mathbf{r}, t)$ and scalar potential $V(\mathbf{r}, t)$, you perform transformations:

$$\mathbf{A}' = \mathbf{A} + \alpha_0 \mathbf{r} \exp\left(-\frac{r^2}{2a^2}\right) \sin \omega t, \tag{2}$$

$$V' = V + \beta_0 \exp\left(-\frac{r^2}{2a^2}\right) \cos \omega t. \tag{3}$$

where a, α_0, β_0 and ω are constants.

9. (6 points) Find the condition under which this is a gauge transformation. Your answer will be an equation involving a, α_0, β_0 and ω.

Solution:
Method 1: The magnetic field is

$$\begin{aligned}\mathbf{B}' = \nabla \times \mathbf{A}' &= \nabla \times \left(\mathbf{A} + \alpha_0 \mathbf{r} \exp\left(-\frac{r^2}{2a^2}\right) \sin \omega t\right) \quad [1\text{pt}] \\ &= \mathbf{B}, \text{because the curl of any radial function is zero. } [1\text{pt}]\end{aligned}$$

So there are no conditions from the magnetic field. The electric field is

$$\begin{aligned}\mathbf{E}' &= -\nabla V' - \frac{\partial \mathbf{A}'}{\partial t} \\ &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} - \nabla\left(\beta_0 \exp\left(-\frac{r^2}{2a^2}\right) \cos \omega t\right) - \frac{\partial}{\partial t}\left(\alpha_0 \mathbf{r} \exp\left(-\frac{r^2}{2a^2}\right) \sin \omega t\right) \quad [1\text{pt}] \\ &= \mathbf{E} - \beta_0 \cos \omega t \nabla\left(\exp\left(-\frac{r^2}{2a^2}\right)\right) - \alpha_0 \omega \mathbf{r} \exp\left(-\frac{r^2}{2a^2}\right) \cos \omega t \quad [1\text{pt}]\end{aligned}$$

Since there are no angular dependencies, $\nabla \ldots = \frac{\partial \ldots}{\partial r} \hat{\mathbf{r}}$, so $\nabla\left(\exp\left(-\frac{r^2}{2a^2}\right)\right) = -\frac{r}{a^2} \exp\left(-\frac{r^2}{2a^2}\right) \hat{\mathbf{r}}$ [1pt]. We find

$$\mathbf{E}' - \mathbf{E} = \left(\alpha_0 \omega - \frac{\beta_0}{a^2}\right) \exp\left(-\frac{r^2}{2a^2}\right) \cos(\omega t)\, \mathbf{r},$$

and thus the required condition is $\beta_0 = \alpha_0 \omega a^2$ [1pt].
Method 2: We derived in class that the fields are invariant under transformations of the form

$$\begin{aligned}\mathbf{A}' &= \mathbf{A} + \nabla \lambda, \\ V' &= V - \frac{\partial \lambda}{\partial t}, \quad [1\text{pt}]\end{aligned}$$

So we need to find λ such that

$$\nabla\lambda = \alpha_0 \mathbf{r} \exp\left(-\frac{r^2}{2a^2}\right) \sin\omega t, \quad [1\text{pt}]$$

$$-\frac{\partial\lambda}{\partial t} = \beta_0 \exp\left(-\frac{r^2}{2a^2}\right) \cos\omega t. \quad [1\text{pt}].$$

The second is trivial to integrate:

$$\lambda = -\frac{\beta_0}{\omega} \exp\left(-\frac{r^2}{2a^2}\right) \sin\omega t + C(\mathbf{r}), \quad [1\text{pt}]$$

where C is independent of t. For the first, integrate using $\nabla\lambda = \frac{\partial\lambda}{\partial r}\hat{\mathbf{r}}$ (spherical coordinates, there are no angular dependencies), giving:

$$\lambda = \alpha_0 \sin\omega t \int r \exp\left(-\frac{r^2}{2a^2}\right) dr = -\alpha_0 a^2 \exp\left(-\frac{r^2}{2a^2}\right) \sin\omega t + D(t), \quad [1\text{pt}]$$

These are equal only if $\beta_0 = \omega\alpha_0 a^2$, providing the required condition [1pt]. (The integration constant can neither depend on radius nor time, so it must be a number and therefore uninteresting.)

Mentioning a constant of integration is not necessary for full credit (though it's better to do it!). Incorrectly leaving an arbitrary function of t or $\mathbf{r}$ in your answer does cost points.

Instead of integrating both equations, you can integrate one and fill in the λ you get into the second equation. Still full credit for a correct answer.

You can also find the condition by starting from the identity $\frac{\partial}{\partial t}\nabla\lambda = \nabla\frac{\partial}{\partial t}$ and inserting your results for $\nabla\lambda$ and $\frac{\partial}{\partial t}$, respectively. This works because the integration constant is not important for the argument.

Mentioning something that would not result in a gauge transformation (e.g. $a = \infty$, which would make the exponentials 1 so the fields become oscillatory) as a second option costs a point.

Incorrect solutions and common mistakes:

- Many students confused vectors and scalars or div and grad. A few derived a form for λ and then said this was somehow the condition.
- It does not suffice to just say that the entire result you get from $\mathbf{E} - \mathbf{E}'$ must be zero, or to stop even earlier. The question instructs you to find an equation involving specific quantities. For full credit, you need to realize that the equality must hold for all r, t, and that $\beta_0 = \omega\alpha_0 a^2$ is the only way to do this.
- Mentioning $a = 0$ as a second possibility shows mathematical creativity, but makes the transformation ill-defined at $r = 0$. No penalty for mentioning it, no extra points either.
- $\alpha_0 = \beta_0 = 0$ is one way to make the transformation a gauge transformation, but far from general. The question asks you to give *the condition*, not *an example*.
- Some creative spirits used Lorenz or Coulomb gauges…

This concludes the test. When you are finished, please:

- Write your **name and student number on every sheet**!
- If you used two sheets, mark them 'Sheet 1/2' and 'Sheet 2/2'. When you hand them in, bind them with **two paperclips** on opposite sides.
- Feed your solutions to the wooden box. Not in the box = not graded.
- Return your formula sheet and *unused* paper. Take this question paper, your cheat sheet, and *used* scratch paper home.