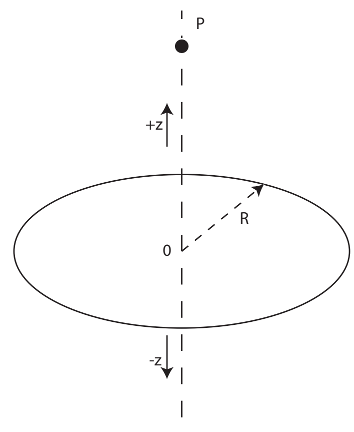
Electricity and Magnetism, Test 1 2023

6 questions, 47 points

> Write your name and student number on each answer sheet. Use of a calculator is allowed. You may make use of 1 A4 (double sided) with handwritten notes and of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\mathbf{A}$ is a vector, $\hat{\boldsymbol{x}}$ is the unit vector in the x-direction, and T is a scalar.
> **In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol**.

1. **Charged loops**.

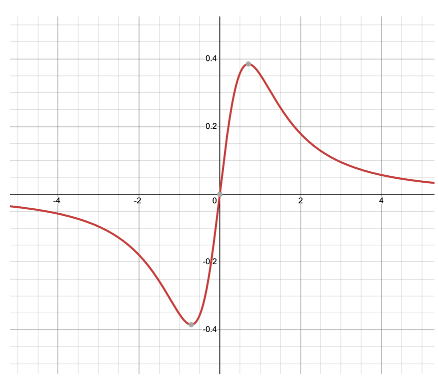


(a) (4 points) Consider a circular loop of radius R, with a uniform line charge $+\lambda$ per unit length. Calculate the electric field at a point P, which is a distance z above the center of the ring.

Answer (a): This is equal to problem 2.5 in the book. From $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda dl}{\mathscr{r}^2}\cos\theta\hat{\mathbf{z}}$, *with* $\mathscr{r}^2 = R^2 + z^2$, $\cos\theta = \frac{z}{\mathscr{r}} = \frac{z}{(R^2+z^2)^{1/2}}$ *and* $\int dl = 2\pi R$, *we find* $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\pi R\lambda z}{(R^2+z^2)^{3/2}}$.

(b) (4 points) Make a qualitative sketch of the electric field magnitude as a function of z, both above and below the loop. Make sure to incorporate the direction of the electric field in its sign.

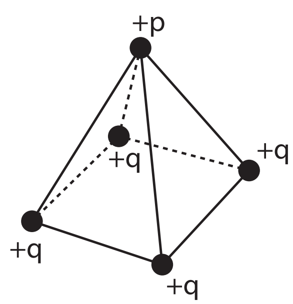
Answer (b): In the plane of the ring the field is zero, it will increase gradually in both directions as you move away from this plane, to reach a maximum before droping again further away from the plane of the ring ($/z^2$). The field is opposite in both sides of the ring, which results in a different sign of the plot of $|\mathbf{E}|$.



2. (6 points) **Charged planes**. Consider two infinite parallel vertical planes that carry uniform charge densities. The plane on the left has charge density σ, the plane on the right has charge density 2σ. Show how you use Gauss's law in this situation, and find the field in each of the three regions: (i) to the left of both, (ii) between them, and (iii) to the right of both.

Answer: This is a variation on Example 2.6. First you need to show how to obtain the field of a single plane with charge density σ*, using the Gaussian pillbox on a surface, as in Example 2.5, to show that* $\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$*. Now you can do Example 2.6, with the field* $\frac{-3\sigma}{2\epsilon_0}$ *in region (i),* $\frac{-\sigma}{2\epsilon_0}$ *in region (ii) and* $\frac{3\sigma}{2\epsilon_0}$ *in region (iii). Here the sign indicates the direction: negative is to the left.*

3. **Charge configuration.** Consider four equal charges (each $+q$) at the corners of square with side length a.



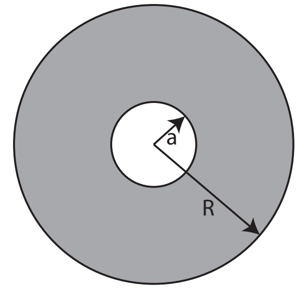
(a) (4 points) What is the total work $W_{1,2,3,4}$ required to place these charges?
Answer (a): $W_{1,2,3,4} = \frac{q^2}{4\pi\epsilon_0}(0 + \frac{1}{a} + (\frac{1}{a} + \frac{1}{\sqrt{2}a}) + (\frac{2}{a} + \frac{1}{\sqrt{2}a})) = \frac{q^2}{4\pi\epsilon_0}\frac{4+\sqrt{2}}{a}$.

(b) (1 point) (+1 bonus point) Now we bring in a fifth charge, with same sign but different magnitude $+p$. We place this charge at a distance a from all of the four other charges, thereby forming a configuration where the charges are at the corners of a pyramid. This takes an additional amount of work W_5. All five charges have the same mass m.
(i) Explain what happens if we were to simultaneously release all charges from their initial positions.
(ii) What is the final velocity (at very large separation) of the charge $+p$? You can express this in terms of $W_{\text{total}} = W_{1,2,3,4} + W_5$, which you do not have to calculate explicitly.

Answer: For (b)(i), the charges will all fly away from each other, since they have the same sign of charge and the coulomb force is repulsive. The charge p travels purely vertically, the four charges q travel sideways and away from p. This gives 1 point.
For (b) (ii), this is actually difficult to calculate. The idea of the question was that W_{total} *is distributed equally over the five charges - but since they are not equivalent, that actually doesn't happen. This would be true for a situation of four charges of equal mass at the corners of a regular tetrahedron, and then the kinetic energy of p would be* $\frac{1}{2}m{v_p}^2 = \frac{1}{4}(W_{total})$ *from which* v_p *follows. An additional complication is the fact that p has a different charge compared to q. For a system of two charges this doesn't matter; two unequal charges exert an equal force on each other. But the repulsion between a pair of charges q is different from that of the* q, p *pairs, and this makes it even harder. This question is converted into a 1 point bonus question, for realising and stating that we cannot calculate the final velocity here based on symmetry arguments. There is no deduction of points for any wrong answers.*

4. **Spherical objects and various materials**.

Consider a massive spherical ball (radius R), that has a small spherical cavity (radius a) carved out as shown in the cut-through figure on the right. The center of the small cavity coincides with the center of the large ball.



(a) (4 points) Assume the material of the object is such that it is (and remains) uniformly charged with density ρ. What is the electric field outside the object? Show how you obtain your answer.

(b) (4 points) Assume now that the material of the object is a conductor, and that it is charged to a total charge Q. What is the electric field outside the object? Show how you obtain your answer.

(c) (4 points) Now a point charge Q is placed inside the middle of the spherical cavity. The material of the object is still conductive, but now without net charge. What is the electric field outside the object? Show how you obtain your answer.

(d) (4 points) Now we consider a conductive object which is similar to the one described in the previous subquestion - but with the center of the spherical cavity offset by a distance b from the center. What is the electric field outside the object? Explain how you obtain your answer.

Answer (a): The superposition principle can be used to subtract the small cavity from the larger cavity. Use Gauss's law with a Gaussian surface around the large solid sphere, which results in $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{\rho}{r^2} d\tau \hat{\mathbf{r}}$. *From this you subtract the field of a sphere with radius a. Alternatively, you just integrate from a to R. The total field is* $\mathbf{E} = \frac{1}{3\epsilon_0}\frac{\rho}{r^2}(R^3 - a^3)\hat{\mathbf{r}}$.
Answer (b): The field outside is just that of a charge Q at the center: $\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$. *The reason is that from Gauss' law outside the conductor you don't have to know where the charges are. Also if they are on the surface of the conductor, the field is indistinguishable from that of charge Q in center.*
Answer (c): the electric field outside is still just that of a charge at the center. The charge in the cavity will be surrounded with negative surface charges on the inside of the conductor. The net charge of the conductor is zero - so on the outside of the conductor we have an equal amount of positive charges. When applying Gauss' law to the whole system, we still just find charge Q - so the field as if a single charge Q is in the center.
Answer (d): still we get the same answer - since the electric field inside the conductor is zero, we can not know from the outside where the cavity and the charge is. So again it looks like a charge Q at the center.

5. **Conceptual questions**. For each subquestion below there are four statements. Identify **all wrong** statements (for each subquestion there is at least one, but there could be more), and **explain on your answer sheet why they are not correct**.

   (a) (4 points) Consider a number of individual charges inside a volume. The flux through the surface of the volume ...

   1. ... is proportional to the total charge inside the volume.
   2. ... depends on the shape of the volume.
   3. ... depends on the surface area of the volume.
   4. ... depends on the sign of individual charges.

   Answer: Gauss's law states that the flux depends on the total charge inside the volume, so statement 1. is correct. The shape (2.) and surface area (3.) do not influence the flux, so these are wrong. The sign of the individual charges matters because it influences the total charge, so (4.) is also correct. Even if the total amount of charge is the same but with opposite sign, the flux is reversed so it is different.

   (b) (4 points) Consider a conductor with a net charge $+q$. The electric field ...

   1. ... everywhere inside the conductor is zero.
   2. ... everywhere outside the conductor is zero.
   3. ... inside and outside the conductor is the gradient of a scalar field.
   4. ... just outside the surface has a direction perpendicular to the conductor's surface normal

   Answer: (1.) is correct, because charges are free to move inside a conductor. If there would be an electric field, there would be a force on the charges, and they would move to the surface. (2.) is not correct - because the conductor is charged, there is an electric field. (3.) is generally correct: this scalar field is called the potential. You can also correctly argue that there has to be a minus sign, and for that reason the statement is wrong. About (4.): the electric field can not have a component along the surface of the conductor - the charges would redistribute to nullify this component, since they are free to move. So statement (4.) is wrong.

   (c) (4 points) Consider the electric potential V.

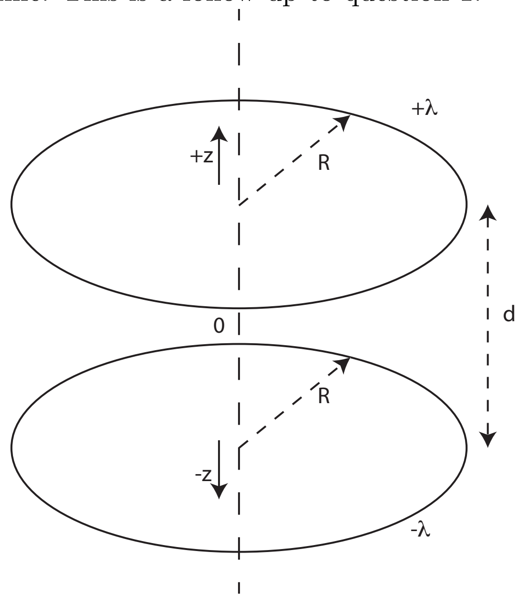
   1. The inside surface and the outside surface of a hollow conductor can be at a different potential.
   2. The path integral of the potential from point a to point b near some charges does not depend on the path taken.
   3. The path integral of the electric field from point a to point b near some charges does not depend on the path taken.
   4. The gradient of the potential between two infinite parallel plates with fixed opposite charges is constant everywhere between the two plates, but this constant value increases when we decrease the distance between the two plates.

   (1.) is wrong - a conductor is an equipotential, also on the inside. (2) is not correct, because the integral of the potential, which is a scalar field, depends on the length of the path, in the case that the potential is constant for example. (3) is correct: the electric field is a conservative force. When point a and b are the same, the total integral is even zero. (4) is not correct: we are considering the gradient of the potential while decreasing the spacing

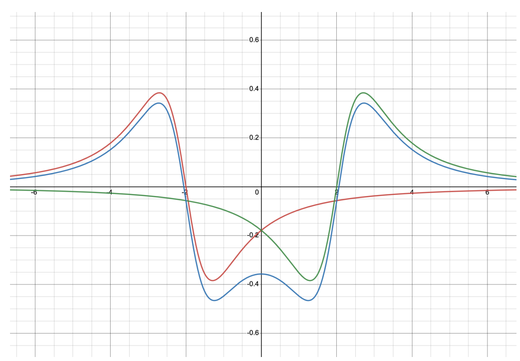
between two plates with fixed charges. For infinite plates the voltage is proportional to d, and the electric field (while we keep the charge constant) is independent of the spacing between the plates. This would not be the case if we consider fringe fields, or the case where the voltage is kept constant.

6. **Bonus question** for 2 extra points, if you have the time. This is a follow-up to question 1.

Consider two circular loops of equal dimensions but with opposite charge, that are parallel to each other, at a distance d. Each has radius R with opposite and uniform line charge $\pm\lambda$ per unit length. The loop at positive z has a positive charge, while the loop at negative z has a negative charge. Make a qualitative sketch of the electric field on the z-axis.



Answer: Two graphs for the two loops are plot in the figure on the right, with opposite sign. Given the positive charge for positive z, the field pointing in the +z direction has been chosen to be positive in this graph. Depending on the distance between the two loops the field in the middle will look slightly different. The sum curve of the two loops is the blue one in the figure.



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